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Study of stochastic linear time-varying systems under non-ergodic optimality criteria and analytical modeling of anomalous diffusions

DISSERTATION SUMMARY

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1 Introduction

Relevance and problem setup. Stochastic linear time–varying systems are widely used to model processes in various applications [1]–[21]. The underlying state equations contain additive disturbances modeled by the increments of the Wiener process with a time-dependent diffusion matrix, resulted from use of a number of research methods and approaches. In biological models, such specificity is a consequence of the implementation of the linear noise approximation, see, for example, [2], [3]. In climatology, time-varying coefficients arise from stochastic averaging, see [4]. The use of diffusion approximations based on convergence to solutions of linear stochastic differential equations (SDE) lead to models of linear stochastic systems as an analytic tool in the field of branching processes [5], queuing theory [6], traffic flows [7], biochemistry [3], cognitive science [8], mathematical insurance [9]. Here the time-varying nature of the coefficients corresponds to time-dependent parameters of the approximated processes, such as inflow/outflow intensities [6] (or impulses [8]), rates of birth/death/immigration [3], [5], insurance premium rate [9], etc. In physical and cognitive research, it is important to take into account time-varying impacts of the environment, which also leads to non-autonomous linear equations of dynamics, see [10]–[12]. The assumption on non-stationary fluctuations along time-varying long-run levels gives motivation to use a class of mean-reverting linear SDEs for modeling economic and financial variables, see [13].

Assuming that control actions are also included in the state equation, we pass to a linear stochastic control system [22], [23], [24, Chapter IX, § 3]. If the goal of control is to keep the system state near a target level, for example, zero, during the planning period, and any deviation results in a loss, while control costs are also considered, then it is natural to choose the objective functional in the integral quadratic form. Since in this case we are talking about the assessment of losses related to points in time, it is important to specify how agents (or subjects of control) account for costs occurred at different times. It is known from behavioral economics, see, for example, [25], that in such a situation one can invoke the concept of time preferences, mathematically expressed using a discount function that depends on a time parameter. Then the corresponding discount function will be incorporated into the cost functional as a multiplier for the current losses. Traditionally, the discount function has the form of a decreasing exponent, i.e. with a constant discount rate, [26, Section 6.1], [27, Section 2.7]. However, various theoretical and practical studies have shown, see [25] for a review, that functions with a time-varying discount rate can be included in the class of discounting functions. The discount rate is defined as the logarithmic derivative of the discount function, taken with a minus sign. «Zero» time preference indicates no discounting. Time preferences with a monotonically decreasing discount function are called «positive». The opposite situation is also possible, the occurrence of «negative» time preferences and an increasing discount function, see [25] and a detailed review in [28], which is typical for loss evaluation and research on large planning horizons when the priority is given to the future, [26, p. 97]. It is also worth noting that a time-increasing cost multiplier may appear as the result of a control system design under extra requirements on system stabilization, see [29], [30], as it was done in engineering applications [31]. It is also obvious that an increasing planning horizon plays an important role in this case. Here we approach the discussion of the main topic, namely, the control problem setup for a linear stochastic system over an infinite time horizon. It is useful to emphasize that the integral quadratic cost functional is also called the risk functional [32], i.e. being able to find a problem solution would also mean minimizing long-term risks. First, let us turn to an approach related to derivation of *average optimal* controls. It is based on a comparison of the mathematical expectations of costs. Here we can apply the concept of *overtaking* optimality on the average over an infinite time horizon, see [26, Section 1.5], which suggests the asymptotic non-positivity of the expected cost difference, in the case when one of the considered controls is *overtaking* optimal. A more common approach is to use the long-run average as an optimality criterion, being a limit superior of the expected losses per unit time, see [22, p. 106], [26, Section 10.2], [27, Section 2.7], [33, Section 5.4].

The long-run average cost criterion can regarded as general for diffusion-type processes, when the control system is time-homogeneous, see [27, Section 3]. The use of the long-run average is based on the idea that the expected value of the functional on the optimal control grows in proportion to the planning horizon length, see [26, Section 10.1]. Obviously, the latter does not hold if the coefficients depend on time, and the optimization of stochastic linear time-varying control systems over an infinite time horizon becomes a very topical issue. Let us outline various important factors impacting the evolution of a linear stochastic control system and control performance assessment in the long run. Such features include time-dependent coefficients of the state equation, for instance, unbounded at infinity or singular, as well as the cost functional with discounting. In particular, we may consider a time-varying diffusion matrix reflecting the degree of uncertainty, as in the cognitive model [11] or the physical model of motion, see [14], [15]. If we turn to the analysis of the deterministic part of the state equation, then examples of unbounded state matrices can be found both in studies on the general theory of linear systems [34], and in some particular models [10], [12], [16], [35]. In addition, a special case here is a non-linear relationship between the internal time of the system and real (physical) time, represented by the time scale transformation [17], [18], implying multiplication of all coefficients by a monotonic function of time, the so-called «scaling», as examples with no controls, one can cite a physical model in [16], a cognitive model of [12], etc. If the rate of time change is random, then we obtain a stochastic time scale used, in particular, in physical [36] and financial [37] applications. When studying control problems on an infinite time interval, it is also worth noting the possibility to consider the so-called two-sided cost functional, when the limits of integration have the opposite sign, which is relevant in the operator-theoretic perspective [38] and models from various applied fields (information transmission [39], engineering [40]). Thus, the issue of setting control problems on increasing time intervals, including the design of optimality criteria over an infinite time horizon, to take into account the above mentioned factors, becomes relevant.

Note that for non-autonomous linear systems, the implementation of linear control laws leads to non-exponential asymptotic stability of the state matrix and give rise to a class of linear time-varying SDEs. Equations of this type are often used in various applications as models of real-life processes, see [1], [2], [10], [13], [15], [16], [19], and reviews in [41], [42]. Along with the case of additive disturbances, i.e. time-varying Ornstein-Uhlenbeck process, it is possible to consider a more general situation of linear SDEs by adding multiplicative noise, as well as observable external, but random inputs to the dynamics. Examples include yield models [43], anomalous diffusions [44], [45], [46]. When analyzing the behavior of solutions of the corresponding equations, the problem naturally arises to estimate their sample path fluctuations near equilibrium, which also allows to reveal the long-term impact of disturbances.

It should be noted that another important application of stochastic linear time-varying systems consists of analytical modeling of processes known as anomalous diffusions. If a linear SDE specifies the velocity, then its integrated solution determines the displacement process. Then the anomalous diffusion is characterized by a non-linear change in time of the second moment of this process, called the mean-squared displacement. Linear growth corresponds to the velocity process in the form of Gaussian «white noise», i.e. the displacement given by Brownian motion, or the integrated standard Ornstein-Uhlenbeck process, the so-called «normal» diffusions. In the case of a nonlinear dependence, subdiffusion and superdiffusion are distinguished. It is important to emphasize that for the velocity processes specified on the basis of linear SDEs, the main characteristics can be written out in a closed form, which makes them an accessible tool for analytical modeling. In this case, it becomes possible to carry out a fairly complete classification determining the types of diffusions, which is also a relevant problem.

State-of-the-art. Turning to the problem of optimality over an infinite time horizon, it is necessary outline the approach associated with the notion of the so-called stochastic optimality in control systems. As it is well known, optimality criteria based on expected values characterize the control performance on the average over the set of all realizations of uncertainty and does not answer the question of what happens if probabilistic statements are used, for example, when trying to compare various sample paths. This is how the concept of stochastic optimality or optimality in terms of probabilistic criteria arises, see [47]-[49]. Here we can recognize the so-called «sensitive» probabilistic criteria, when, by analogy with the notion of average *overtaking* optimality, we consider the minimization of the weighted difference of the costs (almost surely, in probability, in distribution) for different classes of normalizing functions tending asymptotically to zero, see [47], [48], [50], [51]. From this point of view, the linear-quadratic control problem has been studied since the 80s due to the development of a number of probabilistic tools, in particular, martingale methods, see reviews in [49] and [51]. The strongest type of optimality in the probabilistic sense is almost sure optimality or pathwise optimality, when the criterion is minimized with probability 1 [27, Sections 2.7, 3], [47], [48]. A classical example of a probabilistic criterion for controlled diffusion processes is the pathwise ergodic (pathwise average), defined as the limit superior of the ratio of the cost to the planning horizon length [27, Section 2.7], [38]. The crucial assumption when using this criterion is the control system ergodicity, together with a number of other conditions that guarantee the possibility of determining an invariant measure, see [27, Section 3], [38]. To obtain results by the «sensitive» probabilistic criteria [47], [48], time-homogeneity of equations is also required, which makes the methods inapplicable to time-varying systems. At the same time, as it was shown in [51], in the linear-quadratic case and bounded coefficients, one can obtain a closed-form representation for the difference of costs and determine an asymptotic upper bound (almost surely) as a logarithmic function of the planning horizon length, and then involve the pathwise ergodic criterion.

It is also necessary to briefly mention about the methodology of the linear control systems analysis in the long-run. Stabilizability is an important property of a deterministic system ensuring that an infinite-time optimal control problem has a solution. For the bounded coefficient case, it means to achieve exponential stability of trajectories. However, as emphasized in [52], for unbounded or singular matrices such an estimate may be non-informative and a more precise, non-exponential, characteristic of the norm decay of the corresponding fundamental matrix is possible. The issues on non-exponential stabilizability were considered in [53] without addressing the problems of optimal control. Then, using a linear feedback control law, one can obtain a linear SDE describing the evolution of system state. As it is known for the case of bounded coefficients, such an equation is satisfied by a process on a control that is optimal in terms of the long-run averages [22, Section 3.6]. In this case, the control itself is called the optimal stable feedback law, and it can be obtained by passing to the limit in the form of control laws optimal on finite intervals (provided that there exists a «stationary» solution to the Riccati matrix differential equation with no boundary conditions, see [22, Section 3.4]). The study of the asymptotic behavior of the corresponding optimal processes can be carried out along various directions. First, the conditions on the coefficients under which the processes converge to zero are of interest, see [54], [55, Section 4.2,Section 4.3], [56] for the case of bounded coefficients. Secondly, the problem is to obtain a non-random upper bound majorizing sample paths with probability 1, i.e. find the upper function, for special cases see [51], where a logarithmic function was derived, and then replaced by a power function when multiplicative perturbations are added to the dynamics, see [57].

In the area of application of linear SDEs to the anomalous diffusion modeling, various special models have been developed. Thus, analytical modeling of anomalous diffusions can be performed by a time-change in Brownian motion [15] or in the standard Ornstein-Uhlenbeck (OU) process [16], replacing some coefficients of the OU equation to time-dependent ones [11], [12], [58], as well as the use of scaling functions for the basic processes mentioned above, see [20], [59]. It is obvious that the time-varying OU process represents as a generalization of the above models, and it is required to determine the types of anomalous diffusions obtained on its basis. For linear SDEs with multiplicative disturbances, several studies are also being carried out to model anomalous diffusions defined by the SDEs. The processes of scaled geometric Brownian motion [45] or models with power coefficients in the presence of two noise sources [44] are used. The class of linear time-inhomogeneous SDEs, containing various types of noise and stochastic external inputs, can also act as an extension of the already considered special cases. It is worth noting that the topic discussed earlier, related to the study of the asymptotic probabilistic behavior of solutions to such SDEs, has applications to modeling anomalous diffusions. Statements in the form of strong laws of large numbers for integrated processes (i.e., displacement processes) similar to those from [60, Section 5.5] under suitable normalizations make it possible to identify different types of diffusions. In this case, sufficient conditions on the convergence of normalized processes are expressed in terms of the statistical characteristics of the velocity process, determined from the analysis of linear SDEs.

Aim and tasks of the study. The aim of the work is to study optimal control problems over an infinite time horizon for stochastic linear time-varying systems, including the analysis of their asymptotic pathwise behavior, and posterior application to the modeling of anomalous diffusions. In accordance with the goal of the research, the following tasks were identified:

1. to develop a methodology for the analysis of stochastic linear time-varying control systems as the planning horizon tends to infinity, based on finding the so-called optimal stable control law, which is the limiting form of solutions to control problems on finite intervals;

2. to design non-ergodic optimality criteria for control problems over an infinite time horizon, extending the well-known long-run average cost criteria (long-run average and pathwise ergodic) and taking into account various factors affecting the system behavior and control performance assessment in the long run (time-varying diffusion matrix, unbounded matrices in the state equation, discounting in the cost functional and a non-linear time scale);

3. to introduce the notion of criterion efficiency and examine the efficiency of long-run averages taking into consideration a time-varying propperty of the diffusion matrix;

4. to consider the infinite-time control problem for a standard linear control system with bounded coefficients using extended criteria of long-run averages;

5. to obtain the optimal control strategy in the infinite-time control problems for systems with discounting;

6. to find the form of the optimal control law over an infinite time horizon when extending the standard system to the case of inhomogeneous terms in the state equation and the cost;

7. to consider optimal control problems for a linear system with a two-sided quadratic cost under the possibility of the diffusion matrix to be unbounded at infinity;

8. to analyze linear control systems with state matrices unbounded at infinity, including the conditions on the existence of the optimal stable control law;

9. to apply the developed methodology in the study of a linear control system under inversely proportional time-weighting in the cost, i.e. when discount functions having reverse dynamics for different types of losses are included into the cost;

10. to consider linear control systems in the case of multiplying all coefficients by a scaling function, which also corresponds to the incorporation of a non-linear time scale, which can also be of a stochastic nature;

11. to analyze asymptotic behavior of solutions of linear stochastic differential equations with additive noise and non-exponentially stable state matrices in terms of deriving non-random upper bounds on their sample paths;

12. to extend the methodology of analyzing solutions of linear SDEs to the case of a scalar inhomogeneous equation, including correlated additive and multiplicative disturbances;

13. to consider the problem of analytical modeling of anomalous diffusions using linear SDEs for the velocity process.

Research methods. The research methodology includes methods of system analysis, methods of stochastic analysis, methods of optimal control theory, methods of probability theory.

Novelty and reliability. We carry out an analysis of stochastic linear time-varying systems allowing both unbounded growth and singularity of coefficients over time. For the case of linear control systems, an integral quadratic cost functional is also employed, which may include discounting by monotone decreasing or increasing functions, under possibly unbounded discount rate. These features have a key impact on the long-run system behavior and the control performance assessment. In such control systems, the use of the standard long-run average criterion and its pathwise counterpart (pathwise ergodic) may lead to a zero value on a set of controls that have nothing to do with the optimality property, or they might take infinite values on all admissible controls, which gives rise to a new problem of optimality criteria efficiency over an infinite time horizon. We applied a methodology related to finding the so-called optimal stable feedback control law, which is the limiting form (as the planning horizon tends to infinity) of solutions to the corresponding problems posed on finite intervals, i.e. minimization of the expected values of the costs. The use of non-ergodic optimality criteria is proposed, which most precisely accounts for the behavior of the cost and its expected value on the optimal stable control law defined above. As a result, we obtain optimality criteria for the infinite-horizon control problems, extending long-run averages. The generalization is related to the normalizations used to properly accommodate the factors affecting the long-term system behavior (time-varying diffusion matrix, discounting, inhomogeneous terms in the state equation, non-linear time scale, unbounded state matrices). The foundations of optimality criteria design we introduced make it possible to consider control problems over an infinite time horizon for various classes of stochastic linear time-varying systems and to carry out a long-term risk assessment.

The behavior of solutions of linear time-varying stochastic differential equations is analyzed as the time parameter tends to infinity. In particular, such SDEs determine the evolution of the states of stochastic linear control systems under stable linear feedback laws, and are also used to specify the velocity process for anomalous diffusions. It is proposed to consider the non-exponential type of stability of state matrices and to characterize it in terms of the time-dependent rate of change of the upper bound for the norm of the corresponding fundamental matrix. Thus exponential, superexponential and subexponential types of stability are distinguished. To estimate the sample path fluctuations, an upper function is introduced, which depends on the key factors impacting the system evolution and is related to the coefficients of the underlying state equation.

The problem of analytical modeling of anomalous diffusions is studied using the Ornstein-Uhlenbeck process with arbitrary time-varying coefficients under the condition of asymptotic stability related to the state coefficient. In particular, the periodic nature of this function is allowed, which requires the formulation of upper and lower bounds for the corresponding mean-squared displacement, and an precise definition of normal and anomalous diffusion is given. At the same time, the inverse problem of determining the time-dependent stability rate and diffusion coefficient is also solved to reproduce the given function of the mean-squared displacement, making it possible to analyze various classes of models. A probabilistic setup is also proposed, when we compare the displacement process with the upper function known from the law of the iterated logarithm. Thus, the types of diffusions can be identified based on the closeness of the estimates characterizing the sample path fluctuations to the indicators of normal diffusions. For a wide sense linear SDE with correlated multiplicative and additive noises, as well as stochastic external inputs, an approach is used to model anomalous diffusions, involving the previously obtained mean-square estimates of the path evolutions. More precisely, various statements about suitable normalizations for the second moment of the displacement process are used. In this way, anomalous diffusions are revealed in the mean square sense or from a probabilistic setting when compared with a function from the law of the iterated logarithm.

Approbation of the research. The main results of the study were reported at the Research Seminar of the Department of Applied Mathematics MIEM HSE (June 2023), at the International conference «International Conference Computer Data Analysis & Modeling 2022» (Minsk, Belarus, September 6-10 2022), at the Seminar «Veroyatnostnye problemy upravleniya i stohasticheskie modeli v ekonomike, finansah i strahovanii» CEMI RAS (2022, 2021, 2016, 2015, 2014), at the Russian Economic Congress (Moscow, December 21-25 2020; December 19-23 2016), at the All-Russian Conference «Ekonomicheskij rost, resursozavisimost' i social'no-ekonomicheskoe neravenstvo» (Saint Petersburg, October 25-27 2018; November 7-9 2016), at the International conferences «International conference LSA Summer meeting» (Moscow, June 4-5 2018), «Asymptotic Statistics of Stochastic Processes and Applications XI» (Saint Petersburg-Peterhof, July 17-21 2017), «International conference Statistics meets Stochastics 2» (Moscow, June 9-10 2017), «VIII Moscow International Conference on Operations Research ORM 2016» (Moscow, October 17-22 2016), «Workshop on Stochastics, Statistics and Financial Mathematics» (Saint Petersburg-Pushkin, September 3 2016), European Control Conference ECC 2016 (Aalborg, Denmark, June 29-July 1 2016), «2nd Russian-Indian Joint Conference in Statistics and Probability» (Saint Petersburg, May 30-June 3 2016), XVII April International Academic Conference on Economic and Social Development (Moscow, April 19-22 2016), at the International conference «Workshop «Game Theory, Mechanism Design and Market Equilibria» (Saint Petersburg, April 8 2016), Saint Petersburg International Economic Congress SPEC-2016 (Saint Petersburg, March 22 2016), at the International conferences «Bachelier Colloquium on Mathematical Finance and Stochastic

Calculus» (Metabief, France, January 17-24 2016; January 11-18 2015), «Workshop «New Trends in Stochastic Analysis and New Trends in statistical analysis of time series» (Moscow Region, Snegiri, December 7-11 2015), 8th International conference «Upravlenie razvitiem krupnomasshtabnyh sistem MLSD 2015'» (Moscow, September 29-October 1 2015), School on Stochastics and Financial Mathematics-ITIS 2015' (Sochi, September 7-11 2015), at the All-Russian Conference «Sociophysics and Socioengineering» (Moscow, June 8-11 2015), at the International conferences «Second Conference on Stochastics of Environmental and Financial Economics» (Oslo, Norway, April 20-24 2015), «Stochastic Calculus, Martingales and Financial Modeling» (Saint Petersburg, June 29-July 6 2014), at the XII All-Russian Meeting on Control Problems VSPU (Moscow, June 16-19 2014).

List of papers. The results are published in the following papers:

[1*] *Belkina T.A., Palamarchuk E.S.* On stochastic optimality for a linear controller with attenuating disturbances // Automation and Remote Control. 2013. Vol. 74. No. 4 P. 628–641.
https://doi.org/10.1134/S0005117913040061

[2] *Palamarchuk E.S.* Asymptotic behavior of the solution to a linear stochastic differential equation and almost sure optimality for a controlled stochastic process // Computational Mathematics and Mathematical Physics. 2014. Vol. 54. No. 1. P. 83–96.
https://doi.org/10.1134/S0965542514010114

[3*] *Palamarchuk E.S.* Analysis of criteria for long-run average in the problem of stochastic linear regulator // Automation and Remote Control. 2016. Vol. 77. No. 10. P. 1756–1767.
https://doi.org/10.1134/S0005117916100039

[4] *Palamarchuk E.S.* Stabilization of linear stochastic systems with a discount: modeling and estimation of the long-term effects from the application of optimal control strategies // Mathematical Models and Computer Simulations. 2015. Vol. 7. No. 4. P. 381–388.
https://doi.org/10.1134/S2070048215040080

[5] *Palamarchuk E.* On infinite time linear-quadratic Gaussian control of inhomogeneous systems // 2016 European Control Conference (ECC). IEEE, 2016. P. 2477–2482.
https://doi.org/10.1109/ECC.2016.7810662

[6*] *Palamarchuk E.S.* Optimal controller for a nonautonomous linear stochastic system with a two-sided cost functional // Automation and Remote Control. 2020. Vol. 81. No. 1. P. 53–63.
https://doi.org/10.1134/S0005117920010051

[7*] *Palamarchuk E.S.* Optimization of the superstable linear stochastic system applied to the model with extremely impatient agents // Automation and Remote Control. 2018. Vol. 79. No. 3. P. 439–450.

https://doi.org/10.1134/S0005117918030049

[8*] *Palamarchuk E.S.* On the optimal control problem for a linear stochastic system with an unstable state matrix unbounded at infinity // Automation and Remote Control. 2019. Vol. 80. No. 2. P. 250–261.
https://doi.org/10.1134/S0005117919020048

[9] *Palamarchuk E.S.* On optimal stochastic linear quadratic control with inversely proportional time-weighting in the cost // Theory of Probability & Its Applications. 2022. Vol. 67. No. 1. P. 28–43.
https://doi.org/10.1137/S0040585X97T990733

[10*] *Palamarchuk E.S.* Time invariance of optimal control in a stochastic linear controller design with dynamic scaling of coefficients // Journal of Computer and Systems Sciences International. 2021. Vol. 60. No. 2. P. 202–212.
https://doi.org/10.1134/S1064230721020106

[11] *Palamarchuk E.S.* Optimal control for a linear quadratic problem with a stochastic time scale // Automation and remote control. 2021. Vol. 82. No. 5. P. 759–771.
https://doi.org/10.1134/S0005117921050027

[12*] *Palamarchuk E.S.* On the generalization of logarithmic upper function for solution of a linear stochastic differential equation with a nonexponentially stable matrix // Differential Equations. 2018. Vol. 54. No. 2. P. 193–200.
https://doi.org/10.1134/S0012266118020064

[13*] *Palamarchuk E.S.* On asymptotic behavior of solutions of linear inhomogeneous stochastic differential equations with correlated inputs // Differential Equations. 2022. Vol. 58. No. 10. P. 1291–1308.
https://doi.org/10.1134/S00122661220100019

[14*] *Palamarchuk E.S.* An analytic study of the Ornstein—Uhlenbeck process with time-varying coefficients in the modeling of anomalous diffusions // Automation and Remote Control. 2018. Vol. 79. No. 2. P. 289–299.
https://doi.org/10.1134/S000511791802008X

[15] *Palamarchuk E.S.* On upper functions for anomalous diffusions governed by time-varying Ornstein–Uhlenbeck process // Theory of Probability & Its Applications. 2019. Vol. 64. No. 2. P. 209–228.
https://doi.org/10.1137/S0040585X97T989453

* indicates the papers in Journals Q1/Q2 Web of Science/Scopus (9 papers)

The items in the list of published papers are presented in the order corresponding to the summary of the main results of the dissertation concerning issues on stochastic optimal control of linear systems [1] – [11], studies on the state dynamics [12] – [13], analytical modeling of anomalous diffusions [13] – [15].

Author's personal contribution to the development of the problem. The papers [2]–[15] have a single author, the paper [1*] is co-authored by T.A. Belkina.

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Validity of the results. The results are strictly proved mathematical statements.

Theoretical and practical significance. The main provisions of the work contribute to the methodology of the analysis of linear stochastic systems. Various tools are presented for the long-term impact assessment of employed control strategies and anomalous diffusion modeling, which can be used in practical applications.

The main results to be defended.

- A methodology for the analysis of stochastic linear time-varying control systems with an integral quadratic performance index is proposed as the planning horizon tends to infinity. The approach is based on finding the type of the so-called optimal stable control law (OSF) as a linear state feedback. The OSF strategy is a limiting form related to solving problems of minimizing the expected costs on finite intervals and involves a solution of the matrix Riccati differential equation.

- The design of non-ergodic optimality criteria in control problems over an infinite time horizon extending the long-run average cost criteria, is carried out. In this case, the criteria of long-run averages are understood as the usual long-run average, which serves to reveal the optimality property on the average, and the pathwise average, also called the pathwise ergodic, used in optimization with probability 1 (almost surely), i.e. characterizing pathwise optimality. The designed criteria involve information about the factors affecting the long-term system behavior, such as a time-varying diffusion matrix, discounting in the cost, a non-linear time scale, unbounded state matrices in the underlying state dynamics.

- For a standard linear control system with bounded coefficients, under standard conditions of exponential stabilizability and detectability of matrices, it is shown that the control of the OSF form is a solution to the problem with the extended long-run average cost criteria, when the usual cost normalization by the planning horizon length is replaced by the integral of the the diffusion matrix squared norm.

- The notion of efficiency for the optimality criterion over an infinite time horizon is introduced, meaning the positiveness of its value on the control of the OSF form and inefficiency when the value is equal to zero on a set of controls. It is shown that the long-run average cost criteria are inefficient in the case when variances of cumulative disturbances grow slower than the planning horizon length. The proposed criteria of extended long-run averages turn out to be efficient with respect to the factor of the time-varying diffusion matrix.

- Infinite-time control problems for linear systems with discounting are considered. The form of optimal control strategies is found based on the criteria associated with the accumulated discount.

- For a linear control system with bounded coefficients, having inhomogeneous terms in the state equation and the cost (affine components in the state equation and linear state/control dependent terms in the cost), optimality criteria are built up on an infinite time interval, where the normalization represents a measure of the integral deviation of the coefficients in the inhomogeneous part of the control system from a standard deterministic linear-quadratic controller. Conditions on the optimality of the OSF control law according to these criteria are established.

- An analysis of a linear control system with a two-sided cost functional, in which the limits of integration have the opposite sign, is carried out under the assumption of a bounded or monotonically time-increasing norm of the diffusion matrix. Conditions on the optimality of the OSF control law are established when extending long-run averages.

- Linear-quadratic control problems over an infinite time horizon are considered in cases where the state matrix in the underlying equation becomes unbounded at infinity. The cases of superexponentially stable and anti-stable matrices are investigated, the corresponding notions of stability/antistability are introduced, and a characteristic for their rates is proposed. The optimality of the OSF control law was established according to the criterion of the adjusted extended long-run average, where the adjustment is expressed by means of the stability/anti-stability rates.

- A linear control system under inversely proportional time-weighting in the cost and absolutely integrable at infinity matrices related to the state is considered. It is shown that the control of the OSF feedback form is optimal with respect to the criterion of the adjusted extended long-run average and, under a set of conditions on the diffusion matrix, also optimal in the pathwise sense.

- The cases of linear stochastic control systems under dynamic scaling of coefficients are considered, i.e. when multiplying all matrices by a time-varying function, also interpreted as the result of using a non-linear time scale, which can also be of a stcohastic nature. The invariance of the OSF control law, coinciding with the solution of the autonomous linear-quadratic control problem, is established. Conditions are also found under which the stochastic normalization in the extended long-run average criterion can be replaced by a deterministic one without loss of information about the control performance.

- An asymptotic analysis of the behavior of solutions of linear time-varying SDEs under additive disturbances and non-exponentially stable state matrices is carried out. The form of a deterministic upper function is derived, with probability 1 asymptotically majorizing the sample paths of the process.

- For a scalar time-inhomogeneous linear SDE, including correlated additive and multiplicative disturbances, as well as stochastic external inputs, we analyzed the behavior of solutions deriving upper bounds (in the mean square sense and almost surely). A closed form of the mean-square upper bound is obtained as a function of the variances of processes corresponding to solutions of equations that contain only one type of external perturbations. The upper function, as an almost sure estimate on the sample path fluctuations, is determined taking into account the adjustment multiplier as the integral of the squared diffusion coefficient of multiplicative disturbances.

- Linear time-varying SDEs were studied in the direction of analytical modeling of anomalous diffusions. An exact definition of normal and anomalous diffusion is provided by comparing the upper and lower estimates for the mean-squared displacement with the length of the observation horizon. The problem of finding the parameters of the underlying equation to reproduce a given function of mean-squared displacement has been solved. It is shown that in this case the stability rate and the diffusion coefficient must be related by the Riccati equation known from the filtering theory.

- A probabilistic setup is proposed to identify the types of diffusion by comparing the displacement process with the sample path characteristics of diffusions known from the law of the iterated logarithm (i.e., upper functions). The upper functions of displacement processes are obtained in a closed-form under various assumptions on coefficients. The comparison of the results related to the diffusion type identification based on the classification of mean-squared displacements and on the upper functions is performed.

- For a linear SDE with correlated additive and multiplicative disturbances, as well as stochastic external inputs, conditions are found on its coefficients under which the corresponding process defines subdiffusion in the mean-square sense and with respect to upper functions.

2 A summary of the work: main results

2.1 Methodology of stochastic linear control systems analysis in the long-run

We consider models of linear controlled stochastic processes with time-varying coefficients, characterizing the state evolution of systems. All stochastic processes introduced below are defined on the complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$. It is assumed that an n-dimensional stochastic process X_t, $t \geq 0$, is governed by the equation

$$dX_t = A_t X_t dt + B_t U_t dt + G_t dW_t\,, \qquad X_0 = x, \tag{2.1.1}$$

where the initial state x is a non-random; W_t, $t \geq 0$, is a d-dimensional standard Wiener process; U_t, $t \geq 0$, is an admissible control, i.e. k-dimensional stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t = \sigma\{W_s,\ s \leq t\}$, such that (2.1.1) has a solution ($\sigma(\cdot)$ denotes the σ-algebra); A_t, B_t, G_t are time-varying deterministic matrices of appropriate dimensions. Let us denote by $\mathcal{U}$ the set of admissible control. Assume that the system is not deterministic on the whole real time-line, $\int\limits_0^\infty \|G_t\|^2\, dt > 0$ ($\|\cdot\|$ is the Euclidean norm).

For each $T > 0$, as a cost functional we define a random variable (r.v.)

$$J_T(U) = \int_0^T (X_t' Q_t X_t + U_t' R_t U_t)\, dt, \tag{2.1.2}$$

where $U \in \mathcal{U}$ is an admissible control on $[0, T]$; $Q_t \geq 0$, $R_t > 0$, $t \geq 0$, are symmetric positive semidefinite and positive definite matrices respectively. The relation $A > B$ ($A \geq B$) indicates that their difference $A - B$ is positive definite (positive semidefinite), $'$ denotes the transpose.

In the traditional sense, a control U^{*T} is called optimal on interval $[0, T]$ (we further call it average optimal) if

$$EJ_T(U^{*T}) = \inf_{U\in\mathcal{U}} EJ_T(U). \tag{2.1.3}$$

For the model (2.1.1)–(2.1.2), a solution to the problem (2.1.3) is well-known (see, e.g., [22, Theorem 3.9]). The optimal control has the form $U_t^{*T} = -R_t^{-1} B_t' \Pi_t^T X_t^{*T}$, where a symmetric matrix $\Pi_t^T \geq 0$ solves the Riccati equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t' \Pi_t - \Pi_t B_t R_t^{-1} B_t' \Pi_t + Q_t = 0 \tag{2.1.4}$$

with the terminal condition $\Pi_T^T = 0$; X_t^{*T}, $0 \leq t \leq T$, is the optimal process determined by (2.1.1) for $U_t = U_t^{*T}$. In (2.1.4) the dot sign is used to denote the time derivative, the upper case ^{T} means that the solutions are derived for problems under finite T. Assume that there exists $\lim_{T\to\infty} \Pi_t^T = \Pi_t$. Then, passing to the limit in the form of optimal control U_t^{*T}, $T \to \infty$, we obtain a so called *optimal stable feedback control law*

$$U_t^* = -R_t^{-1} B_t' \Pi_t X_t^* \,, \tag{2.1.5}$$

where $\Pi_t \geq 0,\ t \geq 0,$ also satisfies (2.1.4), the process $X_t^*,\ t \geq 0$ is governed by

$$dX_t^* = (A_t - B_t R_t^{-1} B_t' \Pi_t) X_t^* dt + G_t dW_t \,, \quad X_0^* = x \,. \tag{2.1.6}$$

The control U^* is independent of the parameter T, and in a natural way may possess optimality properties over an infinite time horizon with respect to appropriate criteria. Traditionally, see, for example, [26, Section 10.2], the long-run average is used as an optimality criterion for $T \to \infty$ and the problem is being solved

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{T} \to \inf_{U\in\mathcal{U}} . \tag{2.1.7}$$

When optimizing aimed at revealing the property of stochastic optimality (almost surely, a.s.), a pathwise analogue of the long-run average cost criterion is used, pathwise ergodic (pathwise long-run average), see [27], considering the problem

$$\limsup_{T\to\infty} \frac{J_T(U)}{T} \to \inf_{U\in\mathcal{U}} \text{ with probability 1.} \tag{2.1.8}$$

As it is shown below, in the case of the system (2.1.1)–(2.1.2) with time-varying coefficients, the above criteria are inefficient. As a result of the analysis, their various generalizations are proposed for the case of a time-varying diffusion matrix, discounting in the cost, unbounded matrices in the state equation, etc. The approach introduced below for optimality criteria design, $T \to \infty$, is aimed at taking into account the order of change in $EJ_T(U)$ and $J_T(U)$ in a most precise manner when applying the stable feedback control law U^*. It is easy to verify, see [61], [62], [63], that for the functionals $EJ_T(U^*)$ and $J_T(U^*)$ the following representations hold

$$EJ_T(U^*) = x'\Pi_0 x - E[(X_T^*)'\Pi_T X_T^*] + \int_0^T tr(G_t'\Pi_t G_t)\, dt, \tag{2.1.9}$$

$$J_T(U^*) = EJ_T(U^*) + E[(X_T^*)'\Pi_T X_T^*] - [(X_T^*)'\Pi_T X_T^*] + 2\int_0^T (X_t^*)'\Pi_t G_t dW_t \,, \tag{2.1.10}$$

where $tr(\cdot)$ denotes a matrix trace. Then, turning to (2.1.9)–(2.1.10), one can derive the sequence of steps necessary for setting and solving optimal control problems for the system (2.1.1)–(2.1.2), as $T \to \infty$.

1. prove the existence of a matrix $\Pi_t \geq 0$, which is a solution of the Riccati equation (2.1.4), and define the function $p_t > 0,\ t \geq 0$, for which $\limsup\limits_{t\to\infty}\{\|\Pi_t\|/p_t\} < \infty$

2. introduce a normalizing function $\Gamma_T,\ T > 0$, of the form

$$\Gamma_T = \int\limits_0^T p_t \|G_t\|^2\, dt \tag{2.1.11}$$

3. introduce a control problem

$$\limsup_{T\to\infty} \frac{E J_T(U)}{\Gamma_T} \to \inf_{U\in\mathcal{U}} \tag{2.1.12}$$

4. prove the optimality of U^* in the problem (2.1.12), i.e. reveal the average optimality property over an infinite time horizon

5. introduce a control problem

$$\limsup_{T\to\infty} \frac{J_T(U)}{\Gamma_T} \to \inf_{U\in\mathcal{U}} \quad \text{with probability 1} \tag{2.1.13}$$

6. prove the optimality of U^* in the problem (2.1.13), i.e. reveal the pathwise optimality property over an infinite time horizon.

It is obvious that the implementation of the steps 4 and 6 involves finding appropriate conditions on the diffusion matrix G_t, and the step 1 is related to imposing requirements on the coefficients of the deterministic control systems, i.e. the matrices A_t, B_t, Q_t and R_t. Criteria in the problems (2.1.12), (2.1.13) are referred to as *adjusted extended long-run averages* cost criteria. Here, generalization means the change in the standard long-run averages (2.1.7)–(2.1.8) when taking into account the time-varying diffusion matrix, see [61], [64], [65], and the adjustment in the form of multiplication by the function p_t is carried out in order to exhibit the impact of time-varying coefficients A_t, B_t, Q_t and R_t, as was performed in [62], [63], [66], [67]. For systems with discounting (i.e., when the matrices Q_t and R_t depend on a multiplier being the discount function of time), when interpreting the criteria, a particular terminology is used related to the concept of «cumulative discount», see [28], [62], [68], [69], [70].

It is worth emphasizing that the establishing of the optimality property, $T \to \infty$, as a result of solving problems (2.1.7)–(2.1.8), (2.1.12), (2.1.13), is associated with the use of criteria involving cost functionals or their expected values under various normalizations.

Along with this, there is also an approach based on the concept of the so-called *overtaking* optimality. Initially, the concept of *overtaking* optimality arose in the analysis of a number of deterministic models in mathematical economics, see [26, Section 1.4], and later was adapted for controlled diffusion processes in [70]. Average *overtaking* optimality is obtained via a direct comparison of the expected costs on different admissible controls at $T \to \infty$.

Definition 2.1.1 ([64]) *A control $U^* \in \mathcal{U}$ is called average overtaking optimal if for any $\epsilon > 0$ there exists $T_0 > 0$ such that for an arbitrary admissible control $U \in \mathcal{U}$ it holds that*

$$EJ_T(U^*) < EJ_T(U) + \epsilon \quad \text{for any } T > T_0. \tag{2.1.14}$$

Note that the average *overtaking* optimality of the control implies its optimality in the sense of (2.1.7) and (2.1.12).

2.2 Standard linear-quadratic controller with bounded coefficients

Assumption 2.2.1 *The matrices A_t, B_t, Q_t, R_t are bounded, where $R_t \geq \rho I$, $\rho > 0$ is some constant, $t \geq 0$. The pair (A_t, B_t) is (exponentially) stabilizable, the pair (A_t, C_t) is (exponentially) detectable for some matrix C_t, such that $C_t C_t' = Q_t$.*

Definition 2.2.1 ([71]) *The pair of bounded matrices (A_t, B_t) is called (exponentially) stabilizable if there exists a bounded piecewise-continuous matrix K_t such that $A_t + B_t K_t$ is exponentially stable. The pair of bounded matrices (A_t, C_t) is (exponentially) detectable if (A_t', C_t') is (exponentially) stabilizable.*

Definition 2.2.2 ([22]) *A matrix $\mathcal{A}_t$ is exponentially stable if its corresponding fundamental matrix $\Phi(t, s)$ admits the upper bound $\|\Phi(t, s)\| \leq \kappa_0 e^{-\kappa(t-s)}$, $s \leq t$, with some constants $\kappa, \kappa_0 > 0$.*

Recall that the fundamental matrix $\Phi(t, s)$ corresponding to $\mathcal{A}_t$, $t \geq 0$, is defined as a solution to the problem

$$\frac{\partial \Phi(t, s)}{\partial t} = \mathcal{A}_t \Phi(t, s), \qquad \frac{\partial \Phi(t, s)}{\partial s} = -\Phi(t, s)\mathcal{A}_s, \quad \Phi(t, t) = \Phi(s, s) = I, \tag{2.2.1}$$

where I is an identity matrix. As it was discussed in [69] and [71, Theorem 2.2], given Assumption 2.2.1, there exists $\lim_{T \to \infty} \Pi_t^T = \Pi_t$, where the bounded matrix $\Pi_t \geq 0$ satisfies (2.1.4), and the matrix $\mathcal{A}_t = A_t - B_t R_t^{-1} B_t' \Pi_t^T$ is exponentially stable. In addition, see [69], the matrix properties from Assumption 2.2.1 allows to write down an estimate relating the paths of a deterministic linear controller, which is used when comparing the functionals

$J_T(U^*)$ and $J_T(U)$, $T > 0$. More precisely, there is a constant $c_0 > 0$ such that for any pair $(x_t, u_t)_{t \le T}$, satisfying the equation

$$dx_t = A_t x_t dt + B_t u_t dt\,, \quad x_0 = 0\,, \tag{2.2.2}$$

it holds that

$$\|x_T\|^2 + \int_0^T \|x_t\|^2\, dt \le c_0 \int_0^T (x_t' Q_t x_t + u_t' R_t u_t) dt. \tag{2.2.3}$$

The requirement (2.2.3), previously introduced in [51], was also included in the set of sufficient conditions for the existence of optimal controls, $T \to \infty$, in [28], [64], [65], [68]. Further, when presenting the corresponding results [64], [65], [68], Assumption 2.2.1 will be used as the most compact and related to the basic properties of linear control systems.

Let Assumption 2.2.1 hold and the diffusion matrix G_t be bounded, $t \ge 0$. In order to take into account the impact of disturbances on the system when evaluating control performance, we introduced criteria of extended long-run averages in the problems (2.1.12)–(2.1.13) with $\Gamma_T = \int_0^T \|G_t\|^2\, dt$, see [64]. The papers [64], [65] provide results on solving the problems (2.1.12) and (2.1.13), as well as conditions under which the obtained control law U^* has the *overtaking* optimality property on average.

Theorem 2.2.1 ([64][1)],[65][2)]) *Let Assumption 2.2.1 hold and the diffusion matrix G_t is bounded, $t \ge 0$. The the control law U^*, given by (2.1.5)–(2.1.6), is a solution to the problem*

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int_0^T \|G_t\|^2\, dt} \to \inf_{U \in \mathcal{U}}\,, \tag{2.2.4}$$

i.e. it is optimal over an infinite time horizon with respect to the extended long-run average cost criterion. If, in addition, one of the conditions $\int_0^\infty \|G_t\|^2\, dt < \infty$, $\lim_{t\to\infty} \|G_t\| = 0$, is satisfied, then the control U^* is also average overtaking optimal. Moreover, if $\int_0^T \|G_t\|^2\, dt \to \infty$, $T \to \infty$, then U^* is optimal in a pathwise sense, i.e. it solves the problem

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T \|G_t\|^2\, dt} \to \inf_{U \in \mathcal{U}}\,, \quad \text{with probability 1.}$$

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1) See Theorem 1 in *Belkina T.A., Palamarchuk E.S.* On stochastic optimality for a linear controller with attenuating disturbances // Automation and Remote Control. 2013. Vol. 74. No. 4 P. 628–641.

2) See Theorem 2 in *Palamarchuk E.S.* Asymptotic behavior of the solution to a linear stochastic differential equation and almost sure optimality for a controlled stochastic process // Computational Mathematics and Mathematical Physics. 2014. Vol. 54. No. 1. P. 83–96.

Here, for $\Gamma_T = \int\limits_0^T \|G_t\|^2\, dt$ and $\limsup\limits_{T\to\infty} \Gamma_T = \infty$, it holds that $\limsup\limits_{T\to\infty}\{EJ_T(U^*)/\Gamma_T\} =$ $= \limsup\limits_{T\to\infty}\{J_T(U^*)/\Gamma_T\} = \limsup\limits_{T\to\infty}\{\int\limits_0^T tr(G_t'\Pi_t G_t)\, dt)/\int\limits_0^T \|G_t\|^2\, dt\} < \infty$, a.s., see [65]. Thus, the value of the pathwise criterion is also finite (almost surely).

2.3 Analysis of long-run average cost criteria

In Section 2.1 it was noted that the standard criteria of long-run averages in problems (2.1.7) and (2.1.8) do not take into account many factors affecting the long-run evolution of control system. A situation may also arise when the results of applying not only the optimal stable control law U^*, but also a whole set of controls that are in no way related to the optimality property for a finite T, will be indistinguishable by these criteria. In such cases, it is reasonable to emphasize the inefficiency of the employed criteria. We introduce the following definition.

Definition 2.3.1 ([61]) *Let U^* be a stable feedback control law optimal on the average over an infinite time horizon with respect to the criterion $\mathcal{K}$ in system* (2.1.1)–(2.1.2)*. The criterion $\mathcal{K}$ is called*

a) *efficient if* $0 < \limsup\limits_{T\to\infty} E\mathcal{K}_T(U^*) < \infty$ *for* $\limsup\limits_{T\to\infty} EJ_T(U^*) > 0$*;*

b) *inefficient if there exists a set $\mathcal{U}^{\mathcal{E}} \subseteq \mathcal{U}$ such that* $\limsup\limits_{T\to\infty} E\mathcal{K}_T(U^*) = \limsup\limits_{T\to\infty} E\mathcal{K}_T(U^{(\epsilon)}) = 0$ *for any $U^{(\epsilon)} \in \mathcal{U}^{\mathcal{E}}$.*

The efficiency of pathwise criteria is defined similarly, if we replace the expectations in items a) and b) by the values of the functionals themselves. We are interested in the situation when the normalization $\Gamma_T = \int\limits_0^T \|G_t\|^2\, dt$ of the criteria in Section 2.2 grows slower than the planning horizon length T, for example, in a model characterizing the global optimization algorithm [21], or for the anomalous diffusion model in the form of scaled Brownian motion [14].

Assumption 2.3.1

$$\lim_{T\to\infty} \frac{\int\limits_0^T \|G_t\|^2 dt}{T} = 0\,. \tag{2.3.1}$$

Let us construct the set $\mathcal{U}^{\mathcal{E}}$ from Definition 2.3.1. Assume 2.2.1 and let $\kappa_0, \kappa > 0$ be the constants in the exponential upper bound, see Definition 2.2.2. Since B_t is bounded, there exists a $\bar{b} > 0$ such that $\|B_t\| \le \bar{b},\ t \ge 0$. The set of numbers $\mathcal{E}$ is defined as follows

$$\mathcal{E} = \{\epsilon > 0\ :\ \kappa - \epsilon\kappa_0\bar{b} > 0\}\,.$$

For $\epsilon \in \mathcal{E}$, consider the control

$$U_t^{(\epsilon)} = (-R_t^{-1}B_t'\Pi_t + \epsilon I)X_t^{(\epsilon)}\,,$$

where the process $X_t^{(\epsilon)}$, $t \geq 0$, is given by

$$dX_t^{(\epsilon)} = (A_t - B_tR_t^{-1}B_t'\Pi_t + \epsilon B_t)X_t^{(\epsilon)}dt + G_tdW_t\,,\ X_0^{(\epsilon)} = x\,, \tag{2.3.2}$$

and define the set of controls $\mathcal{U}^{\mathcal{E}} = \{U^{(\epsilon)}, \epsilon \in \mathcal{E}\}$. Note that U^* corresponds to $\epsilon = 0$, and, consequently, $U^* \in \mathcal{U}^{\mathcal{E}}$.

Theorem 2.3.1 ([61][3]) *Let Assumption 2.2.1 hold. Then, for any $U^{(\epsilon)} \in \mathcal{U}^{\mathcal{E}}$, there exist $\bar{c}_\epsilon, c_\epsilon > 0$, such that*

$$\limsup_{T\to\infty} \frac{EJ_T(U^{(\epsilon)})}{\int\limits_0^T \|G_t\|^2\,dt} < \bar{c}_\epsilon, \quad \limsup_{T\to\infty} \frac{J_T(U^{(\epsilon)})}{\int\limits_0^T \|G_t\|^2\,dt} < c_\epsilon,\ \text{a.s.}\,,$$

where $\bar{c}_\epsilon$ is a constant, c_ϵ is a r.v. Moreover, $\bar{c}_\epsilon = c_\epsilon$, a.s., if $\int\limits_0^T \|G_t\|^2\,dt \to \infty$, $T \to \infty$. If Assumption 2.3.1 is satisfied, then

$$\lim_{T\to\infty} \frac{EJ_T(U^\epsilon)}{T} = \lim_{T\to\infty} \frac{J_T(U^\epsilon)}{T} = 0\,,\ \text{a.s.}$$

Taking into account the representations (2.1.9) and (2.1.10), one may conclude that the efficiency of extended long-run averages is independent of the specific diffusion matrix. Here, the key role is played by the behavior of the matrix Π_t which is determined on the basis of the coefficients of the deterministic control system. If $\|\Pi_t\| \to 0$, $t \to \infty$, then the extended long-run averages turn out to be inefficient, see [61].

2.4 Inhomogeneous linear control system

This section presents the results of the paper [69] on further generalization of the long-run average cost criteria. In addition to the disturbances, other types of impacts on the control system in the long run are taken into account. We include a time-inhomogeneous affine component in the underlying dynamics, as well as linear state and control dependent terms in the cost functional. An important application of such control systems is the problem of tracking deterministic trajectories, see [72]–[74]. Let us proceed to the setup, where the

[3] See Theorem 1, Theorem 2 in *Palamarchuk E.S.* Analysis of criteria for long-run average in the problem of stochastic linear regulator // Automation and Remote Control. 2016. Vol. 77. No. 10. P. 1756–1767.

previously introduced system (2.1.1)–(2.1.2) becomes a particular case. Assume we are given an n-dimensional controlled stochastic process X_t, $t \geq 0$, governed by

$$dX_t = A_t X_t dt + B_t U_t dt + m_t dt + G_t dW_t\,, \quad X_0 = x\,, \tag{2.4.1}$$

where m_t is a non-random vector, the rest of variables in (2.4.1) are defined similarly to the case of (2.1.1). We also use the notation of $\mathcal{U}$ for the set of admissible controls. The cost over the planning horizon $[0, T]$ is given by

$$J_T(U) = \int_0^T (X_t' Q_t X_t + U_t' R_t U_t + 2q_t' X_t + 2r_t' U_t)\, dt, \tag{2.4.2}$$

where $U \in \mathcal{U}$ is an admissible control on $[0, T]$; $Q_t \geq 0$, $R_t > 0$, $t \geq 0$, are symmetric matrices, q_t, r_t are time-varying vector functions. For the vector functions m_t, r_t, q_t, defining inhomogeneous part in (2.4.1)–(2.4.2), we allow them to be unbounded, as $t \to \infty$. More specifically, we adopt the following assumption.

Assumption 2.4.1 ([69]) *Let* $L_T = \|m_T\|^2 + \|q_T\|^2 + \|r_T\|^2$. *Then*

$$\lim_{T\to\infty} \frac{L_T}{\int\limits_0^T L_t\, dt} = 0\,.$$

By Assumption 2.4.1, possible extreme (exponential) growth of inhomogeneous part is avoided. We also need to assume that the standard requirements concerning system properties are satisfied, see Assumption 2.2.1. Then there exists a vector function $p_t = \int\limits_t^\infty \Phi'(s,t)[\Pi_s(m_s - -B_s R_s^{-1} r_s) + q_s]\, ds,\ t \geq 0,$ satisfying a linear inhomogeneous differential equation

$$\dot{p}_t + \mathcal{A}_t' p_t + \Pi_t(m_t - B_t R_t^{-1} r_t) + q_t = 0\,, \tag{2.4.3}$$

where $\Phi(t,s)$ is the fundamental matrix corresponding to $\mathcal{A}_t = A_t - B_t R^{-1} B_t' \Pi_t,$ the matrix $\Pi_t \geq 0,\ t \geq 0,$ satisfies the Riccati equation (2.1.4). In this case we may define an *optimal stable* feedback law U^* of the form

$$U_t^* = -R_t^{-1}[r_t + B_t'(\Pi_t X_t^* + p_t)]\,, \tag{2.4.4}$$

where the matrix Π_t satisfies (2.1.4), the functions p_t obeys (2.4.3), the process $X_t^*,\ t \geq 0,$ is given by

$$dX_t^* = (A_t - B_t R^{-1} B_t' \Pi_t) X_t^* dt + \mu_t dt + G_t dW_t\,, \quad X_0^* = x\,, \tag{2.4.5}$$

where $\mu_t = m_t - B_t R_t^{-1}(r_t + B_t' p_t)\,.$

As an average optimality criterion over an infinite time horizon in the system (2.4.1)–(2.4.2), it is proposed to use an extension of the long-run average to the case of additional inhomogeneous terms in the state dynamics and the cost functional. Here the idea of designing such a criterion is based on the same considerations as in Section 2.1, i.e. the most precise assessment of the growth order of $EJ_T(U)$ and $J_T(U)$ on the control U^*, as $T \to \infty$:

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int_0^T (\|G_t\|^2 + L_t)\, dt} \to \inf_{U\in\mathcal{U}} . \tag{2.4.6}$$

In the homogeneous case, $L_t \equiv 0$, criterion in (2.4.6) becomes the extended long-run average (see (2.2.4) and Section 2.2), where only the impact of G_t is taken into account. In (2.4.6), the normalizing function represents an integral deviation (over the planning horizon) of the control system (2.4.1)–(2.4.2) from a deterministic linear controller without affine terms ($G_t \equiv 0$, $L_t \equiv 0$).

One of the main results of this section concerns the average optimality of the control U^*, as $T \to \infty$.

Theorem 2.4.1 ([69][4]) *Let Assumptions 2.2.1 and 2.4.1 hold. Then the control law U^*, given by (2.4.4)–(2.4.5), is a solution to (2.4.6).*

Below we state the result on the optimality under the pathwise version of the criterion (2.4.6) considering the problem

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T (\|G_t\|^2 + L_t)\, dt} \to \inf_{U\in\mathcal{U}} \quad \text{with probability } 1 . \tag{2.4.7}$$

Theorem 2.4.2 ([69][5]) *Let the conditions of Theorem 2.4.1 be satisfied and $\int_0^T (\|G_t\|^2 + L_t)\, dt \to \infty$, $T \to \infty$. Then the control law U^*, given by (2.4.4)–(2.4.5), is a solution to (2.4.7). Moreover,*

$$\limsup_{T\to\infty} \frac{J_T(U^*)}{\int_0^T (\|G_t\|^2 + L_t)\, dt} = \limsup_{T\to\infty} \frac{EJ_T(U^*)}{\int_0^T (\|G_t\|^2 + L_t)\, dt} , \tag{2.4.8}$$

holds almost surely.

2.5 Linear control systems with discounting

This section presents a number of results on the analysis of optimal control problems for systems with discounting, when the cost matrices in (2.1.2) have the form $Q_t = f_t Q$,

[4] See Theorem 1 in *Palamarchuk E.* On infinite time linear-quadratic Gaussian control of inhomogeneous systems // 2016 European Control Conference (ECC). IEEE, 2016. P. 2477–2482.

[5] See Theorem 2 in *Palamarchuk E.* On infinite time linear-quadratic Gaussian control of inhomogeneous systems // 2016 European Control Conference (ECC). IEEE, 2016. P. 2477–2482.

$R_t = f_t R$, where $Q \geq 0$, $R > 0$ are constant matrices, f_t is a monotone discount function. Moreover, in the state equation (2.1.1) $A_t = A$, $B_t = B$ are also constant matrices. The cost has a tracking form

$$J_T^{(d)}(U) = \int_0^T f_t[(X_t - \tilde{x}_t)' Q (X_t - \tilde{x}_t) + (U_t - \tilde{u}_t)' R (U_t - \tilde{U}_t)]\, dt\,, \tag{2.5.1}$$

where non-random vectors $\tilde{x}_t$, $\tilde{u}_t$ specify the reference trajectories of state and control. Regarding the discount function, we assume that various conditions are satisfied.

Assumption 2.5.1 ([68]) *The discount function $f_t > 0,\ t \geq 0,\ f_0 = 1$:*
1) *it is monotone, differentiable; if f_t is increasing, then $f_t \to \infty, t \to \infty$, if f_t is decreasing, then $f_t \to 0,\ t \to \infty$;*
2) *the discount rate $\phi_t = -\dot{f}_t / f_t$ is bounded for any $t \geq 0$ ($\dot{}$ is a time derivative), and $\lim\limits_{t \to \infty} \phi_t = c_\phi$, where c_ϕ is a constant.*

Note, see [68], that decreasing f_t ($\phi_t > 0$) corresponds to «positive» time preferences, in case of «negative» time preference f_t increases ($\phi_t < 0$), if $f_t \equiv 1$ ($\phi_t \equiv 0$), then we have «zero» time preferences, see Introduction. Examples of discount functions include standard exponential discounting $f_t = e^{-rt}$, «hyperbolic» $f_t = 1/(1 + \theta t)^{\theta_1/\theta}$, see [75], increasing power $f_t = (1 + t)^k$ [31] or exponential $f_t = e^{rt}$, functions [29] and others ($r, k, \theta, \theta_1 > 0$). We also consider the case of decreasing f_t with unbounded rate at infinity, see [62]. In particular, this discount function property may be related to subjects' highly non-linear time perception or their extreme impatience, see [76] and review in [62].

Assumption 2.5.2 ([62]) *The discount function $f_t > 0,\ t \geq 0,\ f_0 = 1,$ is twice differentiable, decreases monotonically, and is logarithmically concave ($(\ln f_t)'' < 0$). The discount rate $\phi_t = -\dot{f}_t / f_t$ is such that $\phi_t \to \infty,\ t \to \infty$.*

Below we formulate an assumption about the reference path behavior.

Assumption 2.5.3 ([69])

$$\lim_{T \to \infty} \frac{f_T(\|\tilde{x}_T\|^2 + \|\tilde{u}_T\|^2)}{\int_0^T f_t(\|\tilde{x}_t\|^2 + \|\tilde{u}_t\|^2)\, dt} = 0\,.$$

The analysis of control systems with discounting is carried out by change of variables and transformations to systems with a modified diffusion matrix. Then the corresponding *optimal stable* feedback control strategy is defined as

$$U_t^* = -R^{-1} B' (\Pi_t X_t^* + \tilde{p}_t) + \tilde{u}_t\,, \tag{2.5.2}$$

where $\Pi_t \geq 0,\ t \geq 0,$ satisfies the Riccati equation (2.1.4) with $A_t = A - (1/2)\phi_t I,\ B_t = B,$ $Q_t = Q,\ R_t = R,$ the function $\tilde{p}_t$ is given by

$$\tilde{p}_t = \int\limits_t^\infty \frac{f_s}{f_t} \Psi'(s,t)(\Pi_s B\tilde{u}_s - Q\tilde{x}_s)\, ds\,, \tag{2.5.3}$$

where $\Psi(t,s)$ corresponds to the matrix $\tilde{\mathcal{A}}_t = A - BR^{-1}B'\Pi_t,$ the process $X_t^*,\ t \geq 0,$ is governed by

$$dX_t^* = (A - BR^{-1}B'\Pi_t)X_t^* dt + B(\tilde{u}_t - R^{-1}B'\tilde{p}_t)dt + G_t dW_t\,, \quad X_0^* = x\,. \tag{2.5.4}$$

Below we provide statements on the optimality of U^* over an infinite time horizon.

Theorem 2.5.1 **([69][6])** *Let Assumption 2.2.1 hold for $A_t = A,\ B_t = B,\ Q_t = Q,\ R_t = R,$ $G_t = G,$ and Assumption 2.5.1 be satisfied for a non-increasing f_t. Then the control law U^*, given by (2.5.2)–(2.5.4), is a solution to*

$$\limsup_{T\to\infty} \frac{EJ_T^{(d)}(U)}{\int\limits_0^T f_t(\|\tilde{x}_t\|^2 + \|\tilde{u}_t\|^2 + \|G\|^2)\, dt} \to \inf_{U\in\mathcal{U}}\,. \tag{2.5.5}$$

If $\int\limits_0^T f_t(\|\tilde{x}_t\|^2 + \|\tilde{u}_t\|^2 + \|G\|^2)\, dt \to \infty,\ T \to \infty,$ then the control U^* is also a solution to the pathwise control problem

$$\limsup_{T\to\infty} \frac{J_T^{(d)}(U)}{\int\limits_0^T f_t(\|\tilde{x}_t\|^2 + \|\tilde{u}_t\|^2 + \|G\|^2)\, dt} \to \inf_{\tilde{U}\in\mathcal{U}} \quad \text{with probability 1.} \tag{2.5.6}$$

Theorem 2.5.2 **([68][7])** *Let Assumption 2.2.1 hold for $A_t = A - (1/2)\phi_t,\ B_t = B,\ Q_t = Q,$ $R_t = R,\ G_t = G,$ as well as Assumption 2.5.1 be satisfied, and $A\tilde{x}_0 + B\tilde{u}_0 = 0$ for $\tilde{x}_t \equiv x_0,$ $\tilde{u}_t \equiv u_0$. Then the inequality*

$$\limsup_{T\to\infty} \frac{EJ_T^{(d)}(U^*)}{\int\limits_0^T f_t\, dt} \leq \limsup_{T\to\infty} \frac{EJ_T^{(d)}(U)}{\int\limits_0^T f_t\, dt} + c_J\,, \tag{2.5.7}$$

is valid for the control law U^*, given by (2.5.2)–(2.5.4), and any $U \in \mathcal{U}$, with some constant $c_J \geq 0$ (c_J is control-independent). Also, $c_J = 0,$ if in Assumption 2.5.1 the value $c_\phi \geq 0$ and

[6] See Theorem 3 in *Palamarchuk E.* On infinite time linear-quadratic Gaussian control of inhomogeneous systems // 2016 European Control Conference (ECC). IEEE, 2016. P. 2477–2482.

[7] See Theorem 1 in *Palamarchuk E.S.* Stabilization of linear stochastic systems with a discount: modeling and estimation of the long-term effects from the application of optimal control strategies // Mathematical Models and Computer Simulations. 2015. Vol. 7. No. 4. P. 381–388.

$c_J > 0$ for $c_\phi < 0$. Moreover [65], *if $\int\limits_0^T f_t\,dt \to \infty,\ T \to \infty,$ and f_t is non-increasing then U^* is a solution to the pathwise control problem*

$$\limsup_{T\to\infty} \frac{J_T^{(d)}(U)}{\int\limits_0^T f_t\,dt} \to \inf_{U\in\mathcal{U}}, \quad \text{with probability 1.}$$

.

Theorem 2.5.3 ([62][8]) *Let Assumption 2.5.2 hold, and $\tilde{x}_t \equiv 0,\ \tilde{u}_t \equiv 0$. Then the control law U^*, given by (2.5.2)–(2.5.4), is a solution to*

$$\limsup_{T\to\infty} \frac{EJ_T^{(d)}(U)}{\int\limits_0^T (f_t/\phi_t)\|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}. \tag{2.5.8}$$

Note that the criteria in (2.5.5)–(2.5.8) are based on the concept of accumulated discount. In (2.5.8), discounting occurs at a higher rate due to Assumption 2.5.2. For $c_J > 0$, i.e. an increasing exponential function f_t in (2.5.1), the relation (2.5.7) is an analog of δ-optimality known for controlled diffusion processes over an infinite time- horizon, see [77].

2.6 Optimal controller with a two-sided cost functional

In this section we provide results related to the analysis of a linear stochastic control system on the interval $[-T, T]$ for $T \to +\infty$. Such a consideration is motivated by both operator-theoretical aspects of studying linear control systems, see [38], and various areas of applications (information transmission in networks [39], engineering systems [40], etc.). The presentation is based on [78]. The applied optimality criterion is the extended long-run average, while for the diffusion matrix, in contrast to the previously considered case of a standard linear controller from Section 2.2, we allow the matrix to be unbounded at infinity. Let us proceed to the setup, see [78]. Consider a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ and let an n-dimensional stochastic process $X_t,\ t \in \mathbb{R},$ where $\mathbb{R}$ is the set of real numbers, be defined on this space according to the equation

$$dX_t = A_t X_t dt + B_t U_t dt + G_t dW_t\,, \tag{2.6.1}$$

where $A_t,\ B_t$ are bounded time-varying matrices; the disturbances are modeled by the so-called two-sided Wiener process $W_t,\ t \in \mathbb{R},$ defined in an usual way, i.e. $W_t = W_t^{(1)},\ t \geq 0,$

[8] See Theorem 2 in *Palamarchuk E.S.* Optimization of the superstable linear stochastic system applied to the model with extremely impatient agents // Automation and Remote Control. 2018. Vol. 79. No. 3. P. 439–450.

and $W_t = W^{(2)}_{-t}$, $t < 0$, where $W^{(1)}_t$, $W^{(2)}_t$, $t \ge 0$ are two independent d-dimensional Wiener processes, see [79, p. 7]; the set of admissible controls $\mathcal{U}$ consists of k-dimensional square integrable stochastic processes U_t, $t \in \mathbb{R}$, adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, $\mathcal{F}_t = \sigma\{W_s,\ s \le t\}$ ($\sigma(\cdot)$ denotes the σ-algebra) such that there exists a solution to (2.6.1); G_t is the diffusion matrix whose elements satisfy the assumptions below. For the time being, note that the disturbance parameters can be either bounded (e.g., constant $G_t \equiv G$ or fading $\|G_t\| \to 0$), or increasing $\|G_t\| \to \infty$, $t \to \pm\infty$.

For $T > 0$, as a two-sided cost functional on $[-T, T]$, we define a random variable

$$J_{2T}(U) = \int_{-T}^{T} (X_t' Q_t X_t + U_t' R_t U_t)\, dt, \tag{2.6.2}$$

where $U \in \mathcal{U}$ is an admissible control; $Q_t \ge qI$, $R_t \ge \rho I$, $t \in \mathbb{R}$, are bounded symmetric matrices, $q, \rho > 0$ are some constants. The optimal controls over an infinite-time horizon are derived using the extended criteria

$$\limsup_{T\to+\infty} \frac{E J_{2T}(U)}{\int_{-T}^{T} \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}} \tag{2.6.3}$$

and

$$\limsup_{T\to+\infty} \frac{J_{2T}(U)}{\int_{-T}^{T} \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}} \text{ with probability 1.} \tag{2.6.4}$$

Again we note that the extended criteria take into account the evolution of G_t in time. For example, we may have the matrix unbounded at infinity, as in the cognitive model [11], or its singularity, see the case of diffusion in [15]. We formulate the assumptions on the coefficients of (2.6.1)–(2.6.2).

Assumption 2.6.1 *The pair (A_t, B_t) is (exponentially) stabilizable, $t \in \mathbb{R}$.*

The exponential stabilizability of the the pair (A_t, B_t), $t \in \mathbb{R}$, see, e.g. [38], is defined in a similar way to the case $t \ge 0$, see Definition 2.2.1 for $t \in \mathbb{R}$. The next assumption concerns the disturbance parameters, i.e., the matrix G_t, $t \in \mathbb{R}$. We introduce the set $\mathcal{T} = \{-\infty; +\infty; \pm\infty\}$ and use the compact notation $t \to \mathcal{T}$ for any of the cases $t \to -\infty$, $t \to +\infty$ или $t \to \pm\infty$.

Assumption 2.6.2 ([78]) *The diffusion matrix G_t satisfies one of the following conditions:*

1) *G_t is bounded for $t \to \mathcal{T}$;*

2) *$\|G_t\| \to +\infty$, G_t is differentiable, and $d \ln \|G_t\|/dt \to 0$, $t \to \mathcal{T}$.*

Under Assumption 2.6.1, see [38], there exists the control law U^* of the form (2.1.5), $t \in \mathbb{R}$, where a bounded symmetric matrix $\Pi_t \geq \bar{p}I$, $\bar{p} > 0$ is a constant, satisfies the Riccati equation (2.1.4), $t \in \mathbb{R}$. The corresponding process X_t^*, $t \in \mathbb{R}$, is given by (2.1.6) without an initial condition, i.e. obeying

$$dX_t^* = (A_t - B_t R_t^{-1} B_t' \Pi_t) X_t^* dt + G_t dW_t \,. \tag{2.6.5}$$

To analyze problem (2.6.4) in the case $\|G_t\| \to \infty$, $t \to \mathcal{T}$, we need a stronger condition than item 2 of Assumption 2.6.2.

Assumption 2.6.3 ([78]) *Let item 2) of Assumption 2.6.2 hold and additionally,* $d\ln\|G_t\|/dt \cdot \ln|t|(\ln\ln|t| + \ln\ln\|G_t\|) \to 0$, $t \to \mathcal{T}$, *where* $\|G_t\|$ *is a monotone function,* $t \to \mathcal{T}$.

The next theorem establishes the optimality of U^*.

Theorem 2.6.1 ([78][9]) *Let Assumptions 2.6.1 and 2.6.2 hold. Then the control law* U^**, given by (2.1.5),(2.6.5), is a solution to*

$$\limsup_{T\to+\infty} \frac{EJ_{2T}(U)}{\int\limits_{-T}^{T} \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}}, \tag{2.6.6}$$

and we have

$$0 < \limsup_{T\to+\infty} \frac{EJ_{2T}(U^*)}{\int\limits_{-T}^{T} \|G_t\|^2\, dt} = \limsup_{T\to+\infty} \frac{\int\limits_{-T}^{T} tr(G_t'\Pi_t G_t)\, dt}{\int\limits_{-T}^{T} \|G_t\|^2\, dt} < \infty\,. \tag{2.6.7}$$

If Assumption 2.6.3 is satisfied and $\int\limits_{-T}^{T} \|G_t\|^2\, dt \to \infty$, $T \to +\infty$, *then the control* U^* *also solves the pathwise control problem (2.6.4).*

According to the result of Theorem 2.6.1, $\limsup\limits_{T\to+\infty}\{EJ_{2T}(U)/(\int\limits_{-T}^{T} \|G_t\|^2\, dt)\} \geq J^*$, where $J^* > 0$ is a constant. In the case $\|G_t\| \to \infty$, $t \to \mathcal{T}$, the irrelevance of the long-run average cost criterion also becomes obvious, since $\limsup\limits_{T\to+\infty}\{EJ_{2T}(U)/(2T)\} = +\infty$, for any admissible control $U \in \mathcal{U}$.

[9] See Theorem 1, Theorem 2 in *Palamarchuk E.S.* Optimal controller for a nonautonomous linear stochastic system with a two-sided cost functional // Automation and Remote Control. 2020. Vol. 81. No. 1. P. 53–63.

2.7 Optimal control under state matrices unbounded at infinity

This section presents results on analysis of a linear stochastic control system under the assumption that the state matrix A_t is unbounded at infinity, i.e. when $\|A_t\| \to \infty,\ t \to \infty$. For example, such a property of A_t may arise in the systems obtained as the result of linearization [80, Section 5.4] or when considering models of certain random processes in various fields such as physical, cognitive science [10], [12], [35], studying the general theory of linear systems [34], including discounting by unbounded rate, see references in [62] and [63]. First, it is worth noting that the matrices A_t with $\|A_t\| \to \infty,\ t \to \infty,$ may possess asymptotic stability/instability properties that significantly differ from the standard exponential notions. Let us provide the necessary definitions.

Definition 2.7.1 ([62]) *The matrix $\mathcal{A}_t$ is called superexponentially stable with the rate δ_t (or δ_t-superexponentially stable) if there exists a function $\delta_t > 0,\ t \ge 0,\quad \delta_t \to \infty,\ t \to \infty$ such $\limsup\limits_{t\to\infty} (\|\mathcal{A}_t\|/\delta_t) < \infty,\ \ \|\Phi(t,s)\| \le \kappa \exp\{-\int\limits_s^t \delta_v\, dv\}\,,\ \ s \le t\,,$ for some $\kappa > 0$, at that $\Phi(t,s)$ is the fundamental matrix corresponding to $\mathcal{A}_t$.*

Let us turn to the main characteristics of the matrix instability. In this case, an asymptotically unbounded increase in the norm of the fundamental matrix is possible. In order to clarify the nature of the instability, the notion of antistability from the theory of operators is used, see, for example, [81, p. 11].

Definition 2.7.2 ([63]) *The matrix $\tilde{\mathcal{A}}_t$ is called superexponenially anti-stable with the rate δ_t if the matrix $\mathcal{A}_t = -\tilde{\mathcal{A}}_t'$ is superexponenially stable with the rate δ_t.*

It is also natural to call superexponentially stable matrices superstable, and anti-stable matrices super unstable. In this section, the cases of superexponentially stable and superexponentially anti-stable matrices A_t are considered. In both situations, the control systems we study do not satisfy the standard assumptions of Section 2.2. According to the *Algorithm of Analysis*, p. 18, it is required to establish the existence of a solution to the Riccati equation (2.1.4), determine its upper bound (i. 1), and then design (i. 4) the corresponding optimality criterion (2.1.12), adjusting the extended long-run average (2.2.4). We also analyze the conditions for the average *overtaking* optimality (see Definition 2.1.1).

Assumption 2.7.1 ([62]) *The matrix A_t is superexponentially stable with the rate δ_t, at that δ_t is a nondecreasing differentiable function, $t \ge 0$; the matrices B_t, Q_t, R_t are bounded, $t \ge 0,\ \ R_t \ge \rho I,$ where $\rho > 0$ is some constant.*

Assumption 2.7.2 ([63]) *The matrix A_t is superexponentially anti-stable with the rate δ_t, at that δ_t is a nondecreasing differentiable function, $t \geq 0$, and $\lim_{t\to\infty}(\dot{\delta}_t/\delta_t^2) = 0$. The bounded matrix B_t is such that $B_t B_t' \geq bI$ $t \geq 0$, where $b > 0$ is some constant. In the cost (2.1.2), the matrices $Q_t \geq qI$, $R_t \geq \rho I$, $t \geq 0$, where $q, \rho > 0$ are some constants.*

The next lemma establishes the existence of solutions to (2.1.4) in the case of A_t satisfying Assumptions 2.7.1 or 2.7.2.

Lemma 2.7.1 ([62][10],[63][11]) *Let Assumption 2.7.1 or Assumption 2.7.2 be satisfied. Then, there exists an absolutely continuous function Π_t, $t \geq 0$, with values in the set of positive semidefinite symmetric matrices satisfying the Riccati equation (2.1.4), and such that the matrix $A_t - B_t R_t^{-1} B_t' \Pi_t$ is $\tilde{\delta}_t$-superexponentially stable with $\tilde{\delta}_t = \lambda\delta_t$, where δ_t is the stability/instability rate of A_t, λ is a positive constant. Moreover, the relation $\limsup_{t\to\infty}(\|\Pi_t\|/p_t) < \infty$ is valid as well, where the function $p_t = 1/\delta_t$ for a superstable A_t, and $p_t = \delta_t$ for a super unstable A_t.*

In addition to Assumption 2.7.2, we also introduce a technical condition.

Assumption 2.7.3 ([63])

$$\lim_{T\to\infty} \frac{\delta_T \|G_T\|^2}{\int_0^T \delta_t \|G_t\|^2\,dt}\delta_T = 0\,. \tag{2.7.1}$$

The following statement characterizes the average optimality of the control law U^* over an infinite time horizon.

Theorem 2.7.1 ([62][12],[63][13]) *Let Assumption 2.7.1 hold or Assumptions 2.7.2,2.7.3 be satisfied. Then the control law U_t^*, given by (2.1.5)–(2.1.6), is a solution to*

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int_0^T p_t \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}\,, \tag{2.7.2}$$

[10] See Lemma in *Palamarchuk E.S.* Optimization of the superstable linear stochastic system applied to the model with extremely impatient agents // Automation and Remote Control. 2018. Vol. 79. No. 3. P. 439–450.

[11] See Lemma in *Palamarchuk E.S.* On the optimal control problem for a linear stochastic system with an unstable state matrix unbounded at infinity // Automation and Remote Control. 2019. Vol. 80. No. 2. P. 250–261

[12] See Theorem 1 in *Palamarchuk E.S.* Optimization of the superstable linear stochastic system applied to the model with extremely impatient agents // Automation and Remote Control. 2018. Vol. 79. No. 3. P. 439–450.

[13] See Theorem in *Palamarchuk E.S.* On the optimal control problem for a linear stochastic system with an unstable state matrix unbounded at infinity // Automation and Remote Control. 2019. Vol. 80. No. 2. P. 250–261

where the function $p_t = 1/\delta_t$ *for a superstable* A_t*, and* $p_t = \delta_t$ *for a super unstable* A_t*. Moreover, if for a superstable matrix* A_t*, we have* $\|G_t\|/\delta_t^2 \to 0,\ t \to \infty,$ *or* $\|G_t\|\delta_t \to 0,$ $t \to \infty,$ *when the matrix* A_t *is super unstable, then control* U^* *is also average overtaking optimal over an infinite time horizon.*

The criterion in (2.7.2) can be called the adjusted extended long-run average, while the adjustment is made by decreasing or increasing the sub-integral function, depending on the cases of superstable or super unstable matrices A_t under consideration.

2.8 Optimal controller under inversely proportional time-weighting in the cost

In this section we present the main results on the analysis of a stochastic process similar to the controlled Brownian motion model [67]. More precisely, the state matrix A_t in the equation (2.1.1) is absolutely integrable at infinity. In particular, such systems appear in physical [14], [82] and financial [83] applications. Here we assume the mutually reverse timing assessment of losses related to the state and control, when the priority is set on the control costs. The value of losses due to state deviation decreases over time, while for the control costs, on the contrary, it increases. For each $T > 0$, the cost is given by

$$J_T(U) = \int_0^T (\frac{1}{\beta_t} X_t' Q X_t + \beta_t U_t' R U_t)\, dt, \tag{2.8.1}$$

where $U \in \mathcal{U}$ is an admissible control on $[0, T]$; $Q \geq 0,\ R > 0$ are symmetric matrices; $\beta_t > 0,\ t \geq 0,$ is a function setting the priority of different types of costs at time t. Here β_t increases sufficiently fast. More precisely, the parameters of the control system (2.1.1)–(2.1.2), (2.8.1) satisfy the following assumption.

Assumption 2.8.1 ([67]) *The state matrix* A_t *in* (2.1.1) *is such that* $\int_0^\infty \|A_t\|\, dt < \infty$ *. The function* β_t *in the cost* (2.8.1) *satisfies the conditions* $\beta_t > 0,\ t \geq 0,\ \beta_t \to \infty$ *as* $t \to \infty$*, and* $\int_0^\infty \frac{1}{\beta_t}\, dt < \infty$ *.*

When analyzing the linear control system, $T \to \infty$, we use the steps from *Algorithm of Analysis*, p. 18. Because the cost (2.8.1) containts both unbounded in time (i.e. $\beta_t R$) and singular (i.e. $(1/\beta_t)Q$) cost matrices, it is required to establish the existence of a solution to the Riccati equation (2.1.4)

Lemma 2.8.1 (**[67]**[14]) *Let Assumption 2.8.1 hold. Then there exists a function* $\Pi_t,\ t \geq 0$, *with values in the set of nonnegative definite symmetric matrices, which satisfy the differential Riccati equation (2.1.4) for* $Q_t = 1/\beta_t,\ R_t = \beta_t R$, *and is such that* $\limsup\limits_{t\to\infty}\{\|\Pi_t\|/p_t\} < \infty$, *where* $p_t = \int\limits_t^\infty \frac{1}{\beta_t}\,dt$. *If, in addition* $Q > 0$, *then* $\liminf\limits_{t\to\infty}\{\|\Pi_t\|/p_t\} > 0$.

Therefore, one can define U^* as (2.1.5)–(2.1.6) and then use the adjusted extended long-run averages (2.1.12), (2.1.13) to reveal the optimality properties of U^*, $T \to \infty$. As it was shown in [67], the sample-path properties of the optimal process $X_t^*\ t \geq 0$, are be close to those of a time-changed Wiener process.

Assumption 2.8.2 (**[67]**) *The following relations hold for* $p_t = \int\limits_t^\infty \frac{1}{\beta_s}\,ds$ *and the diffusion matrix* G_t*:*

1) $\int\limits_0^T p_t\|G_t\|^2\,dt \to \infty\,,\ T \to \infty$;
2) $\limsup\limits_{t\to\infty}\{\beta_t\|G_t\|^2 p_t^2\} < \infty$.

Theorem 2.8.1 (**[67]**[15]) *Let Assumption 2.8.1 hold. Then the control* U^**, given by (2.1.5)–(2.1.6), is a solution to*

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int\limits_0^T p_t\|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}} \tag{2.8.2}$$

where $p_t = \int\limits_t^\infty \frac{1}{\beta_s}\,ds$, *and no restriction on the diffusion matrix* G_t *is needed. If Assumption 2.8.2 is satisfied, then the control* U^* *also solves the pathwise control problem*

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int\limits_0^T p_t\|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}} \text{ with probability } 1\,. \tag{2.8.3}$$

2.9 On time invariance of optimal control law for a class of linear-quadratic regulators

This section presents results on the analysis of the linear stochastic control system (2.1.1)–(2.1.2) for special cases of time-dependence of coefficients. The situations of multiplication of all matrices by the same function of time (i.e. dynamic scaling of parameters, see [84]) are

[14] See Lemma 1 in *Palamarchuk E.S.* On optimal stochastic linear quadratic control with inversely proportional time-weighting in the cost // Theory of Probability & Its Applications. 2022. Vol. 67. No. 1. P. 28–43.

[15] See Theorem 1, Theorem 2 in *Palamarchuk E.S.* On optimal stochastic linear quadratic control with inversely proportional time-weighting in the cost // Theory of Probability & Its Applications. 2022. Vol. 67. No. 1. P. 28–43.

considered. A stochastic process can also act as this function, then the corresponding control system arises due to the incorporation of a stochastic time scale into the analysis, see [85]. It turns out that under such assumptions the optimal control U^* is invariant with respect to the multiplier and the form of U^* coincides with the optimal strategy known for autonomous systems. Assume that on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ we are given a scalar stochastic process $\alpha_t,\ t \geq 0,$ having continuous and positive sample-paths with probability 1. Then the stochastic time scale is defined as an almost surely increasing process $\tau_t = \int\limits_0^t \alpha_v\, dv,\ t \geq 0,$ or, in the differential form,

$$d\tau_t = \alpha_t\, dt, \quad \tau_0 = 0\,. \tag{2.9.1}$$

Various processes can be considered as α_t, $t \geq 0$, for details see [85]. The process $\tau_t,\ t \geq 0,$ is called «internal» time in contrast to physical or real time t.

Assumption 2.9.1 ([85]) *A stochastic process $\alpha_t > 0,\ t \geq 0,$ defining a time scale in (2.9.1), has continuous (a.s. and in mean square) sample paths with $\int\limits_0^t \alpha_v\, dv \to \infty,\ t \to \infty,$ a.s.*

The case of the deterministic function α_t was considered in [84]. As it was shown in [85], incorporation of the stochastic time scale τ_t into the control system $(Y_\tau(\bar{U}), J_\mathcal{T}(\bar{U}))$, known as an autonomous stochastic linear-quadratic controller, $dY_\tau = AY_\tau d\tau + B\bar{U}_\tau d\tau + Gd\hat{W}_\tau,\ J_\mathcal{T}(\bar{U}) = \int\limits_0^\mathcal{T} (Y_\tau' Q Y_\tau + \bar{U}_\tau' R \bar{U}_\tau)\, d\tau,$ leads to the state equation (2.1.1) and the cost (2.1.2) with random coefficients:

$$dX_t = \alpha_t A X_t dt + \alpha_t B U_t dt + \sqrt{\alpha_t} G dW_t\,, \qquad X_0 = x, \tag{2.9.2}$$

$$J_T^{(\alpha)}(U) = \int\limits_0^T \alpha_t (X_t' Q X_t + U_t' R U_t)\, dt, \tag{2.9.3}$$

where the admissible controls $U_t,\ t \geq 0,$ are $\bar{\mathcal{F}}_t$-adapted process, $\bar{\mathcal{F}}_t = \sigma\{W_s, \alpha_s,\ s \leq t\}$ such that (2.9.2) admits a solution. We denote the set of admissible controls by $\mathcal{U}$. Previously, linear systems of the form (2.9.2) with random coefficients and no controls were studied in the areas of physics [36], finance [37] and mechanics [86], for deterministic α_t – in cognitive research, see [12], and econometric modeling [17]–[18], [87]. As $T \to \infty$, we consider the following optimal control problems

$$\limsup_{T\to\infty} \frac{E J_T^{(\alpha)}(U)}{E\left(\int\limits_0^T \alpha_t\, dt\right)} \to \inf_{U\in\mathcal{U}}, \quad \limsup_{T\to\infty} \frac{J_T^{(\alpha)}(U)}{\int\limits_0^T \alpha_t\, dt} \to \inf_{U\in\mathcal{U}} \quad \text{with probability 1.} \tag{2.9.4}$$

It is useful to note that in internal time (i.e. not considering (2.9.1)), problems (2.9.4) would have the form of control problems under long-run averages. The main result on the existence of a solution to problems (2.9.4), as a time-invariant control law $U^* = -R^{-1}B'\bar{\Pi}X^*$, where $\bar{\Pi} \geq 0$ is a solution of the algebraic Riccati equation, is given in the following statement.

Theorem 2.9.1 ([85][16],[84][17]) *Let Assumption 2.2.1 hold for $A_t = A$, $B_t = B$, $Q_t = Q$, $R_t = R$, and Assumption 2.9.1 is satisfied. Then the control U^*, given by (2.1.5)–(2.1.6), solves the problems (2.9.4). We also have*

$$\lim_{T\to\infty} \frac{EJ_T^{(\alpha)}(U^*)}{E\left(\int\limits_0^T \alpha_t\,dt\right)} = \lim_{T\to\infty} \frac{J_T^{(\alpha)}(U^*)}{\int\limits_0^T \alpha_t\,dt} = tr(G'\bar{\Pi}G) \quad \text{a.s.}\,,$$

where a symmetric matrix $\bar{\Pi} \geq 0$ is a solution to the algebraic Riccati equation $A'\bar{\Pi} + \bar{\Pi}A - \bar{\Pi}BR^{-1}B'\bar{\Pi} + Q = 0$.

In the following remark we describe the possibility of transition to non-random normalizations and the form of the corresponding control problems, see [85]. Various examples of stochastic time scales known from applications are also provided in [85].

Remark 1 ([85]) 1. Suppose that for a stochastic process $\alpha_t,\ t \geq 0$, we have $\limsup\limits_{T\to\infty}\{\int\limits_0^T \alpha_t\,dt/\Gamma_T^{(+)}\} = c^{(+)} > 0$ or $\liminf\limits_{T\to\infty}\{\int\limits_0^T \alpha_t\,dt/\Gamma_T^{(-)}\} = c^{(-)} > 0$ with probability 1; $\Gamma_T^{(+)}$, $\Gamma_T^{(-)}$ are positive non-random functions, and $c^{(+)}$, $c^{(-)}$ are some constants. Then, instead of (2.9.4), we can consider the problems

$$\limsup_{T\to\infty} \frac{J_T^{(\alpha)}(U)}{\Gamma_T^{(+)}} \to \inf_{U\in\mathcal{U}} \quad \text{or} \quad \liminf_{T\to\infty} \frac{J_T^{(\alpha)}(U)}{\Gamma_T^{(-)}} \to \inf_{U\in\mathcal{U}}\,.$$

In this case, the values of the criteria on the optimal control U^* will be, respectively, $\limsup\limits_{T\to\infty}\{J_T^{(\alpha)}(U)/\Gamma_T^{(+)}\} = c^{(+)}tr(G'\bar{\Pi}G)$ and $\liminf\limits_{T\to\infty}\{J_T^{(\alpha)}(U)/\Gamma_T^{(-)}\} = c^{(-)}tr(G'\bar{\Pi}G)$.

2. Let $T^{-1}\int\limits_0^T \alpha_t \to \bar{\alpha}$ a.s. and let $T^{-1}\int\limits_0^T E\alpha_t \to E\bar{\alpha},\ T\to\infty,$ where $\bar{\alpha} > 0$ is some random variable. Then (2.9.4) are replaced by problems with criteria given by the long-run averages:

$$\limsup_{T\to\infty} \frac{EJ_T^{(\alpha)}(U)}{T} \to \inf_{U\in\mathcal{U}} \quad \text{and} \quad \limsup_{T\to\infty} \frac{J_T^{(\alpha)}(U)}{T} \to \inf_{U\in\mathcal{U}}\,,$$

however, here $\lim\limits_{T\to\infty}\{T^{-1}EJ_T^{(\alpha)}(U^*)\} = (E\bar{\alpha})tr(G'\bar{\Pi}G)$ и $\lim\limits_{T\to\infty}\{T^{-1}J_T^{(\alpha)}(U^*)\} = \bar{\alpha}\,tr(G'\bar{\Pi}G)$, i.e. the deterministic normalization leads to a difference between the values of the two criteria on U^*, one of the long-run averages will be a random variable.

[16] See Theorem in *Palamarchuk E.S.* Optimal control for a linear quadratic problem with a stochastic time scale // Automation and remote control. 2021. Vol. 82. No. 5. P. 759–771.

[17] See Theorem 2 in *Palamarchuk E.S.* Time invariance of optimal control in a stochastic linear controller design with dynamic scaling of coefficients // Journal of Computer and Systems Sciences International. 2021. Vol. 60. No. 2. P. 202–212.

2.10 Analysis of asymptotic behavior of solutions of linear stochastic systems

In this part of the work, we summarize our results on the analysis of the asymptotic behavior of solutions to linear stochastic differential equations (SDEs) with time-varying coefficients. The studies carried out, see [41] and [88], were aimed at finding upper functions majorizing the sample paths of the process with a probability 1, as the time tends to infinity. We consider cases of additive perturbations, see [41], i.e. the processes of type (2.1.6), and the possibility of adding multiplicative noise, which immediately expands the range of applications, see [88].

2.10.1 Additive disturbance case

First, we describe the class of linear SDEs under consideration, which are processes of the form (2.1.6) in the case of a non-exponentially stable matrix $\mathcal{A}_t$, possibly including cases of an unbounded or singular diffusion matrix G_t, $t \to \infty$. It is assumed that an n-dimensional stochastic Z_t, $t \geq 0$, is governed by a linear SDE

$$dZ_t = \mathcal{A}_t Z_t dt + G_t dW_t\,, \qquad Z_0 = z, \tag{2.10.1}$$

where z is a non-random initial state; W_t, $t \geq 0$, is a d-dimensional standard Wiener process; $\mathcal{A}_t$, G_t, $t \geq 0$, are (non-random) matrices of appropriate dimensions such that there exists a solution to (2.10.1). Here we also assume that $\int\limits_0^\infty \|G_t\|^2\,dt > 0$. As it was noted earlier, we consider the case when the matrix $\mathcal{A}_t$ has a stability property that extends the common notion of exponential stability. In Section 2.7 we already introduced the definition of one of these types of stability, superexponential, see Definition 2.7.1. In the general case, the non-exponential type of stability is characterized by the rate $\delta_t > 0$ with a detailed description in the following definition.

Definition 2.10.1 ([41]) *A matrix $\mathcal{A}_t$ is called stable with a rate $\delta_t > 0$ (or δ_t-stable) if*

(i) $\limsup\limits_{t\to\infty} (\|\mathcal{A}_t\|/\delta_t) < \infty$;

(ii) *there exists $\kappa > 0$ such that*

$$\|\Phi(t,s)\| \leq \kappa \exp\{-\int\limits_s^t \delta_v\,dv\}\,, \qquad s \leq t\,,$$

where $\Phi(t,s)$ is the fundamental matrix corresponding to $\mathcal{A}_t$.

(iii) $\int\limits_0^t \delta_s\,ds \to \infty,\ t \to \infty$.

Note that the exponential stability corresponds to $\delta_t \equiv \kappa_1$ ($\kappa_1 > 0$ is a constant). For $\delta_t \to 0$, $t \to \infty$, we have a weaker notion of subexponential stability, see [66], if $\delta_t \to \infty$, $t \to \infty$, the type of stability is stronger, so called superexponential (see the work [89], devoted to the analysis of nonlinear scalar differential equations, for relevant terminology). It is worth noting that (2.10.1) belongs to the class of equations defining time-varying Ornstein-Uhlenbeck processes. Study of the asymptotic behavior of solutions to (2.10.1) is motivated by the widespread use of such processes in various applications, see the review in [41] and also Section 2.11.1 on anomalous diffusion modeling. Regarding the coefficients of (2.10.1), we assume that the following requirement is met.

Assumption 2.10.1 ([41]) *The matrix $\mathcal{A}_t$ is stable with the rate δ_t, the diffusion matrix G_t satisfies $\limsup\limits_{t\to\infty}(\|G_t\|^2/\delta_t)<\infty$.*

Assumption 2.10.1 implies that $\limsup\limits_{t\to\infty} E\|Z_t\|^2<\infty$. As mentioned earlier, when analyzing the asymptotic behavior of solutions linear to SDEs, the well-known approach is used, which consists in deriving upper functions of stochastic processes.

Definition 2.10.2 ([41]) *A deterministic function $h_t>0$ defines the upper function of a scalar stochastic process $\bar Z_t$, $t\ge 0$ if the following inequality*

$$\limsup_{t\to\infty}\frac{\bar Z_t}{h_t}<\bar c<\infty\,, \tag{2.10.2}$$

holds with probability 1, where $\bar c>0$ is a non-random constant. For example, from the law of the iterated logarithm for the Wiener process, [90, Theorem 8, p. 91], function $h_t=\sqrt{t\ln\ln t}$ if $\bar Z_t=\|W_t\|$. If $h_t\to 0$, then $\limsup\limits_{t\to\infty}\bar Z_t\le 0$ with probability 1. Also, having the form h_t, one can determine the normalization $\tilde\Gamma_T$, under which $\bar Z_t/\tilde\Gamma_t\to 0$, a.s., $t\to\infty$. In this case, we assume that $\bar Z_t=\|Z_t\|^2$, and then we generalize the well-known logarithmic estimate $h_t=\ln t$, previously obtained for the equation (2.1.6) with bounded coefficients, see [51]. For some positive constant $\gamma<1/2$, under conditions of Assumption 2.10.1, we define a bounded function d_t in the form

$$d_t=\int\limits_0^t \exp\{-2\gamma\int\limits_s^t\delta_v\,dv\}\|G_s\|^2\,ds\,. \tag{2.10.3}$$

The main result is the following theorem.

Theorem 2.10.1 ([41][18)]) *Let Assumption 2.10.1 hold true. Then the upper function h_t for the process $\bar Z_t=\|Z_t\|^2$ has the form*

a)

$$h_t=d_t\ln\left(\int\limits_0^t\delta_v\,dv\right)\,, \tag{2.10.4}$$

[18)] See Theorem 1 in *Palamarchuk E.S.* On the generalization of logarithmic upper function for solution of a linear stochastic differential equation with a nonexponentially stable matrix // Differential Equations. 2018. Vol. 54. No. 2. P. 193–200.

if function $d_t \exp\{2\gamma \int\limits_0^t \delta_v\, dv\} \to \infty,\ t \to \infty$;

and

b)

$$h_t = \exp\{-2\tilde{\alpha}\gamma \int\limits_0^t \delta_v\, dv\}, \tag{2.10.5}$$

if function $d_t \exp\{2\gamma \int\limits_0^t \delta_v\, dv\}$ *is bounded. The constants* $0 < \tilde{\alpha} < 1,\ 0 < \gamma < 1/2,$ *with* d_t *being defined by* (2.10.3).

The statement of Theorem 2.10.1 implies that, for any upper function h_t, the relation $c_2 h_t^{(1)} \le h_t \le c_1 h_t^{(0)}$ holds with some constants $c_1, c_2 > 0$, where $h_t^{(0)} = \ln(\int\limits_0^t \delta_v\, dv)$, $h_t^{(1)} = \exp\{-\beta \int\limits_0^t \delta_v\, dv\}$ for some constant $0 < \beta < 1$.

2.10.2 Case of correlated additive and multiplicative disturbances

In this section, we consider the general situation of a scalar linear SDE, when the underlying dynamics is affected by multiplicative disturbances together with additive ones, as well as external inputs in the form of a stochastic process, see [88]. Suppose that, on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ with filtration $\{\mathcal{F}_t\}_{t\ge0}$, we are given a scalar process Z_t, $t \ge 0$, , that is a solution of a linear SDE

$$dZ_t = a_t Z_t dt + \tilde{f}_t dt + G_t dW_t + \sigma_t Z_t dw_t, \tag{2.10.6}$$

with the non-random initial condition $Z_0 = z$; a_t, G_t, σ_t are piecewise-continuous deterministic functions of time; $\tilde{f}_t,\ t \ge 0,$ is an $\mathcal{F}_t$-adapted stochastic process with the property $E\int\limits_0^t \tilde{f}_s^2\, ds < \infty$, $t \ge 0$; W_t, w_t, $t \ge 0$ are correlated $\mathcal{F}_t$-adapted scalar Wiener processes, i.e. $dW_t dw_t = \rho dt$, where ρ is a constant, $-1 \le \rho \le 1$.

Equation (2.10.6) is an equation with time-varying coefficients satisfying the assumptions stated below. Here we note that the situations of both unboundedness and singularity of parameters are allowed as $t \to \infty$. Equations of this type are widely used for modeling in various fields of applications, see [43], [44] and the references in [88], [91].

Assumption 2.10.2 ([88]) *There exists a monotone deterministic function* $\delta_t,\ \delta_t > 0, t \ge 0,$ *such that*

a) $\int\limits_0^t \delta_v\, dv \to \infty,\ t \to \infty,\ \limsup\limits_{t\to\infty}\{(G_t^2 + \sigma_t^2)/\delta_t\} < \infty$

and moreover, the function $\bar{\Phi}(t,s) = \exp\left(\int\limits_s^t a_v\, dv\right)$ *satisfies the inequalities*

b)

$$\kappa_2 \exp\left(-2\bar\kappa \int\limits_s^t \delta_v\, dv\right) \le \bar\Phi^2(t,s) \le \kappa_1 \exp\left(-2 \int\limits_s^t \delta_v\, dv\right), \quad s \le t\,, \tag{2.10.7}$$

c)

$$\bar\Phi^2(t,s) \exp\left(\int\limits_s^t \sigma_v^2\, dv\right) \le \kappa_3 \exp\left(-2\kappa \int\limits_s^t \delta_v\, dv\right), \quad s \le t, \tag{2.10.8}$$

with some constants $\kappa_i,\ \kappa_i > 0$ *(*$i = 1,2,3$*),* $\bar\kappa$ *and* κ *are such that* $\bar\kappa \ge 1,\ 0 < \kappa \le 1$*.*

The conditions in item a) and item b) have the same meaning as the conditions from the previously introduced Definition 2.10.1. If $\tilde f_t \equiv 0,$ then the presence of (2.10.8) and item a) ensure that EZ_t^2 is bounded, $t \ge 0$. Obviously, one of the important questions in studying (2.10.6) is the analysis of its solutions as the time parameter grows, in particular, the possibility of Z_t tending to zero is of interest. As in Section 2.10.1, such an analysis is carried out by deriving upper estimates in some probabilistic sense, as functions of coefficients (2.10.6). More precisely, the problem is to find non-negative functions h_t and $\bar h_t$, $t \ge 0$ such that

$$\limsup_{t\to\infty} \frac{EZ_t^2}{\bar h_t} < \infty\,,$$

and

$$\limsup_{t\to\infty} \frac{Z_t^2}{h_t} < \infty\,, \quad \text{with probability 1.} \tag{2.10.9}$$

If the form of h_t and $\bar h_t$ is known, then one can find conditions on the coefficients under which $EZ_t^2 \to 0$, we have a mean square convergence, or $Z_t^2 \to 0$ with probability 1, which means that the process tends to zero a.s., as $t \to \infty$. Previously, the problem of finding the functions h_t and $\bar h_t$ was studied for some special cases of Eq.(2.10.6), see [41], [55, Section 4.2, p. 117]–[57], [91].

First, we present a result on the form of the majorizing function $\bar h_t,\ t \ge 0,$ for estimating the process in the mean square.

Lemma 2.10.1 ([88][19]) *Let Assumption 2.10.2 be true.*
Then one has the estimate $\limsup\limits_{t\to\infty}\{EZ_t^2/\bar h_t\} < \infty,$ *where the function* $\bar h_t$ *is given in the form*

$$\bar h_t = \exp\{-2\kappa(1-\lambda)\int\limits_0^t \delta_v\, dv\} z^2 + \int\limits_0^t \exp\{-2\kappa(1-\lambda)\int\limits_s^t \delta_v\, dv\}\left(G_s^2 + \frac{E\tilde f_s^2}{\delta_s}\right) ds, \tag{2.10.10}$$

for any constant $\lambda,$ *such that* $0 < \lambda < 1,$ *with the constant* $\kappa,\ 0 < \kappa \le 1,$ *being taken from the condition* (2.10.8)*.*

[19] See Lemma 2 in *Palamarchuk E.S.* On asymptotic behavior of solutions of linear inhomogeneous stochastic differential equations with correlated inputs // Differential Equations. 2022. Vol. 58. No. 10. P. 1291–1308.

The appearance of the constant λ, which reduces the rate of the function in (2.10.10), is due to the presence of a nonzero correlation $\rho \neq 0$ between the Wiener processes governing the additive and multiplicative disturbances in (2.10.6) as well as the contribution of the external inputs $\tilde{f}_t$ in the dynamics (2.10.6).

Next, a number of notations are introduced. Let ϵ be a real number and set

$$\mathcal{N}_t(\epsilon) = \int_0^t \exp\{2\bar{\kappa} \int_0^s \delta_v\, dv + (1+\epsilon) \int_0^s \sigma_v^2\, dv\} G_s^2\, ds, \tag{2.10.11}$$

where $\bar{\kappa} \geq 1$ is the constant in condition (2.10.7).

We also define an $\alpha > 0$ as follows

$$\alpha = \frac{2}{1 + (1 - \kappa/\bar{\kappa})^{-1}} + \beta, \tag{2.10.12}$$

where $\beta > 0$ is an arbitrarily small number and the constants $\bar{\kappa}$ and κ are taken from (2.10.7) and (2.10.8) in Assumption 2.10.2. Then we have the following result.

Theorem 2.10.2 ([88][20]) *Let Assumption 2.10.2 be true.*
Then $\limsup_{t\to\infty}\{Z_t^2/h_t\} < \infty$, *a.s. for the function* h_t

$$h_t = h_t^{(0)} + \int_0^t \exp\{-2(1-\lambda_2) \int_s^t \delta_v\, dv\} \left(G_s^2 + \frac{\tilde{f}_s^2}{\delta_s}\right) ds \left(\int_0^t \sigma_v^2\, dv\right)^{\alpha}, \tag{2.10.13}$$

where the function $h_t^{(0)}$ *is defined by* $h_t^{(0)} = \exp\{-2\int_0^t \delta_v\, dv - 2(1-\lambda_0)\int_0^t \sigma_v^2\, dv\} z^2 +$
$+ \int_0^t \exp\{-2(1-\lambda_1)\int_0^t \delta_v\, dv\} G_s^2\, ds \left(\int_0^t \sigma_v^2\, dv\right)^{\alpha} \ln\left(\int_0^t \delta_v\, dv\right)$, *if the condition* $\mathcal{N}_t(\epsilon) \to \infty$, $t \to \infty$, *holds for any* $\epsilon > 0$, *and by the formula* $h_t^{(0)} = \exp\{-2\int_0^t \delta_v\, dv - 2(1-\lambda_0)\int_0^t \sigma_v^2\, dv\}(1+z^2)$, *if* $\mathcal{N}_\infty(\epsilon) < \infty$ *for some* $\epsilon > 0$.

Here the function $\mathcal{N}_t(\epsilon)$ *is defined in* (2.10.11), *and* λ_i *are arbitrary constants,* $0 < \lambda_i < 1$, $i = 0, 1, 2$. *The number* α *is defined in* (2.10.12) *if* $\limsup_{t\to\infty}\{\sigma_t^2/\delta_t\} > 0$, *and* $\alpha > 0$ *is an arbitrary small number if* $\limsup_{t\to\infty}\{\sigma_t^2/\delta_t\} = 0$. *Moreover, one can set* $\lambda_1 = 0$ *if* $\int_0^\infty \sigma_t^2\, dt < \infty$.

Comparing the result obtained in Theorem 2.10.2 with the estimate derived in the paper [41] and Section 2.10.1 for the case of additive disturbances, one can notice that the new factors included in Eq. (2.10.6) impacted the form of the corresponding estimate (2.10.13).

[20] See Theorem 1 in *Palamarchuk E.S.* On asymptotic behavior of solutions of linear inhomogeneous stochastic differential equations with correlated inputs // Differential Equations. 2022. Vol. 58. No. 10. P. 1291–1308.

The presence of multiplicative noise increases the upper bound in proportion to the value of $(\int_0^t \sigma_v^2\, dv)^\alpha$, along with the assumption about correlation $\rho \neq 0$ and stochastic external inputs $\tilde{f}_t$, which also implies the result in [57].

2.11 Analytical modeling of anomalous diffusions

This section explores the use of scalar linear SDEs (2.10.1) and (2.10.6) to model processes known as anomalous diffusions. Anomalous diffusion means that the mean-squared displacement corresponding to the velocity process given by (2.10.1) or (2.10.6) has a non-linear order (power, logarithmic, etc.). Linear growth corresponds to the displacement given by Brownian motion, the so-called «normal» diffusion. Various aspects of modeling in detail and a literature review are provided in [42], here we focus on the description of the main results. It should be noted that we employ both the well-known approach based on mean-squared displacements, see [42], [88], and the new probabilistic setup, see [92], using the notion of upper functions to characterize anomalous diffusions when compared with the upper function from the iterated log law [90].

2.11.1 Modeling by a time-varying Ornstein-Uhlenbeck process

Let us introduce into consideration the model, which is used subsequently for the analytical specification of anomalous diffusions. Suppose that a scalar stochastic process Z_t, $t \geq 0$, is a time-varying Ornstein-Uhlenbeck process, governed by a non-autonomous linear SDE

$$dZ_t = a_t Z_t dt + \sigma_t dW_t\,, \qquad Z_0 = z, \tag{2.11.1}$$

where the initial state z is a non-random; W_t, $t \geq 0$, is standard Wiener process; a_t, σ_t, $t \geq 0$, are piecewise-continuous time-varying function.

We assume that the function a_t, $t \geq 0$, guarantees stability with rate δ_t for solutions of the corresponding deterministic equation, see items (i)–(iii) of Definition 2.10.1. The item (i) implies that $\Phi(t,s) \geq \kappa_0 \exp\{-\int_s^t \bar{\kappa}\delta_v\, dv\}$, $s \leq t$, for some constants $\kappa_0, \bar{\kappa} > 0$. Assuming that the initial position is zero, we define the displacement process

$$Y_T = \int_0^T Z_t\, dt\,, \quad T \geq 0\,, \tag{2.11.2}$$

and the mean-squared displacement as

$$D_T = E\left(\int_0^T Z_t\, dt\right)^2 . \tag{2.11.3}$$

Further formula (2.11.3) is transformed, see [42], to

$$D_T = 2\int_0^T \int_0^t \Phi(t,s) EZ_s^2 \, ds \, dt \,, \tag{2.11.4}$$

where

$$EZ_t^2 = \Phi^2(t,0) z^2 + \int_0^t \Phi^2(t,s)\sigma_s^2 \, ds \,. \tag{2.11.5}$$

In what follows, the notation $\sim$ introduced in the next definition is used.

Definition 2.11.1 ([42]) *The notation* $f_t \sim g_t$ *means that* $0 < \lim\limits_{t\to\infty}(f_t/g_t) < \infty$ *holds for two non-negative scalar functions* f_t, g_t.

It is interesting to study the behavior of D_T for $T \to \infty$. If for some nondecreasing function $\tilde{D}_T > 0$ it holds that $\tilde{D}_T \sim D_T$ (see Definition 2.11.1), this function $\tilde{D}_T$ will also characterize the order of change in the mean-squared displacement D_T. Note that, see [42], the mean-squared displacement D_T can be estimated as

$$\kappa_0^3 D_T^{(1)} \le D_T \le \kappa^3 D_T^{(2)} \,, \tag{2.11.6}$$

where $D_T^{(1)}$ and $D_T^{(2)}$ are the mean-squared displacements defined on the basis of (2.11.1) for the cases $a_t = -\bar{\kappa}\delta_t$ and $a_t = -\delta_t$, $\kappa_0, \kappa, \bar{\kappa} > 0$ are the corresponding constants, see (ii) in Definition 2.10.1. For these situations, we have also introduced separate notations for second moment functions (see (2.11.5)):

$$m_t^{(1)} = \exp\{-2\int_0^t \bar{\kappa}\delta_v \, dv\} z^2 + \int_0^t \exp\{-2\int_s^t \bar{\kappa}\delta_v \, dv\}\sigma_s^2 \, ds \,, \tag{2.11.7}$$

$$m_t^{(2)} = \exp\{-2\int_0^t \delta_v \, dv\} z^2 + \int_0^t \exp\{-2\int_s^t \delta_v \, dv\}\sigma_s^2 \, ds \,. \tag{2.11.8}$$

Next we formulate the definition of an anomalous diffusion

Definition 2.11.2 ([42]) *Let* $d_1 = \liminf\limits_{t\to\infty}(D_T/T)$ *and* $d_2 = \limsup\limits_{t\to\infty}(D_T/T)$. *If* $0 < d_1 \le d_2 < \infty$, *that the diffusion is called normal, otherwise it is anomalous: for* $d_2 = 0$, *a subdiffusion; for* $d_1 = \infty$, *a superdiffusion.*

In the following statement we show conditions that let us classify diffusion (2.11.1) depending on the characteristics of processes that define mean-squared displacements $D_T^{(1)}$ and $D_T^{(2)}$.

Theorem 2.11.1 ([42][21]) *Let* $d^{(1)} = \liminf_{t\to\infty} (m_t^{(1)}/\delta_t)$ *and* $d^{(2)} = \limsup_{t\to\infty} (m_t^{(2)}/\delta_t)$, *where* $m_t^{(1)}$ *and* $m_t^{(2)}$ *have been defined in* (2.11.7) *and* (2.11.8). *Then the following diffusion types are possible:*

1) *for* $0 < d^{(1)} \le d^{(2)} < \infty$ *- normal diffusion;*

2) *for* $d^{(2)} = 0$ *- subdiffusion;*

3) *for* $d^{(1)} = \infty$ *- superdiffusion.*

Let us consider an inverse problem. Suppose that we know the function D_T, $T \ge 0$, and the problem is to find $a_t = -\delta_t$ and σ_t of Eqs. (2.11.1) for which the corresponding $\tilde{D}_T$ would be $\tilde{D}_T \sim D_T$. It turns out that this problem always has a solution under the natural condition that the mean-squared displacement function increases monotonically. We use the notations $\dot{D}_t$, $\ddot{D}_t$, $\dddot{D}_t$ for the first, second and third derivatives of D_t respectively.

Theorem 2.11.2 ([42][22]) *Let* D_t *be a three times differentiable function, and* $\dot{D}_t > 0$, $t \ge 0$. *Then there exists a pair of functions* (δ_t, σ_t^2), *where* $\delta_t > 0$ *defines the stability rate, and* $\sigma_t^2 > 0$, *related by*

$$\dot{\delta}_t + \frac{3\ddot{D}_t}{\dot{D}_t}\delta_t + 2\delta_t^2 + \frac{\dddot{D}_t}{\dot{D}_t} = \frac{2\sigma_t^2}{\dot{D}_t}, \quad \delta_0 = \bar{\delta}, \tag{2.11.9}$$

($\bar{\delta} > 0$ *is an arbitrary initial condition*), *such that for the displacement process*

$$\tilde{D}_T = 2\int_0^T \exp\{-\int_0^t \delta_v\, dv\} \int_0^t \exp\{\int_0^s \delta_v\, dv\} m_s^{(2)}\, ds\, dt, \tag{2.11.10}$$

it holds that $\tilde{D}_T \sim D_T$. *Here, in* (2.11.10) *the function* $m_t^{(2)}$ *is given by* (2.11.8) *with* $z = 0$.

2.11.2 On upper functions for anomalous diffusions governed by a time-varying Ornstein-Uhlenbeck process

In this section we present the results on the analysis of diffusion types comparing upper functions of displacement processes, see [92] and Definition 2.10.2. In the equation (2.11.1), we assume that $a_t = -\delta_t$ and $z = 0$. Let us make an assumption about the change of the stability rate function δ_t.

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[21] See Theorem 1 in *Palamarchuk E.S.* An analytic study of the Ornstein–Uhlenbeck process with time-varying coefficients in the modeling of anomalous diffusions // Automation and Remote Control. 2018. Vol. 79. No. 2. P. 289–299.

[22] See Theorem 2 in *Palamarchuk E.S.* An analytic study of the Ornstein–Uhlenbeck process with time-varying coefficients in the modeling of anomalous diffusions // Automation and Remote Control. 2018. Vol. 79. No. 2. P. 289–299.

Assumption 2.11.1 ([92]) *The stability rate δ_t is a monotone differentiable function, $t \geq 0$, and the function $\phi_t = \dot{\delta}_t/\delta_t^2$ satisfies at least one of the two conditions*

$$\lim_{t\to\infty} \phi_t = \hat{\kappa}\,, \qquad \lim_{t\to\infty} (1/\phi_t) = \tilde{\kappa}\,, \quad \textit{where } \hat{\kappa}, \tilde{\kappa} \textit{ are non-positive constants} \tag{2.11.11}$$

$$\textit{and, if in (2.11.11)} \quad \hat{\kappa} = \tilde{\kappa} = -1, \ \textit{then} \quad \Big|\int\limits_0^\infty \Big(\frac{1}{t+1} - \delta_t\Big)\,dt\Big| < \infty\,.$$

Note that the cases $\hat{\kappa}, \tilde{\kappa} > 0$ are not considered, because in these cases we have $\delta_t < 0$, $t \geq 0$, that is, in this setting it is clear that the above assumptions about δ_t (see (2.11.1)) are violated because of the unstable coefficient in the underlying equation.

The form of the upper functions is obtained by direct integration of the velocity process. Let us introduce the function B_t by

$$B_t = -\int\limits_t^\infty \exp\{-\textstyle\int_0^s \delta_v\,dv\}\,ds \quad \text{if } \displaystyle\int\limits_0^\infty \exp\{-\textstyle\int_0^t \delta_v\,dv\}\,dt < \infty \tag{2.11.12}$$

and

$$B_t = \int\limits_0^t \exp\{-\textstyle\int_0^s \delta_v\,dv\}\,ds \quad \text{if } \displaystyle\int\limits_0^t \exp\{-\textstyle\int_0^s \delta_v\,dv\}\,ds \to \infty\,,\ t\to\infty\,. \tag{2.11.13}$$

In what follows, by g_T we denote any monotone function with the following properties

$$g_T > 0,\ T \geq 0, \quad g_T \to \infty,\ T \to \infty\,. \tag{2.11.14}$$

Then the following result holds true.

Theorem 2.11.3 ([92][23]) *Let*

$$M_T^{(1)} = \int\limits_0^T \exp\{2\textstyle\int_0^t \delta_v\,dv\}\sigma_t^2\,dt\,, \quad M_T^{(2)} = \displaystyle\int\limits_0^T B_t^2 \exp\{2\textstyle\int_0^t \delta_v\,dv\}\sigma_t^2\,dt\,, \tag{2.11.15}$$

where the function B_t is defined in (2.11.12)–(2.11.13). Then the upper function h_T of the process $\bar{Z}_T = |Y_T|$ is of the following form:

1) *If* $M_\infty^{(1)} < \infty$, $M_\infty^{(2)} < \infty$, *then*

a) $Y_T \to Y_\infty = -\int\limits_0^\infty B_t \exp\{\int_0^t \delta_v\,dv\}\sigma_t\,dW_t$ *a.s.,* $T\to\infty$, *when* $\int\limits_0^\infty \exp\{-\int_0^t \delta_v\,dv\}\,dt < \infty$;

b) $h_T \sim g_T|B_T|$, *in the case* $\int\limits_0^t \exp\{-\int_0^s \delta_v\,dv\}\,ds \to \infty$, $t\to\infty$;

2) *If* $M_\infty^{(1)} < \infty$, $M_T^{(2)} \to \infty$, $T\to\infty$, *then* $h_T \sim |B_T|\sqrt{\ln\ln|B_T|}$;

[23] See Lemma 3 in *Palamarchuk E.S.* On upper functions for anomalous diffusions governed by time-varying Ornstein–Uhlenbeck process // Theory of Probability & Its Applications. 2019. Vol. 64. No. 2. P. 209–228.

3) *If* $M_T^{(1)} \to \infty$, $T \to \infty$, $M_\infty^{(2)} < \infty$, *then* $h_T \sim |B_T| \sqrt{M_T^{(1)} \ln\ln M_T^{(1)}} + g_T$;

4) *If* $M_T^{(1)} \to \infty$, $M_T^{(2)} \to \infty$, $T \to \infty$, *then*

$$h_T \sim |B_T| \sqrt{M_T^{(1)} \ln\ln M_T^{(1)}} + \sqrt{M_T^{(2)} \ln\ln M_T^{(2)}},$$

here g_T *is the function with properties* (2.11.14).

In [92] we also investigated the correspondence between the types of diffusions defined on the basis of mean-squared displacement (see Definition 2.11.2) and using upper functions. We extend the approach to identifying the type of diffusion by comparing its characteristics with the known properties of normal diffusion, in the case under consideration, the upper function $h_T \sim \sqrt{T \ln\ln T}$, which leads to the formulation of the following definition.

Definition 2.11.3 ([92]) *Assume that we know* h_T*, which is the upper function of the displacement process* Y_T *(see Theorem* 2.11.3*). Let* $\bar{d}_1 = \liminf_{t\to\infty} (h_T/\sqrt{T \ln\ln T})$ *and* $\bar{d}_2 = \limsup_{t\to\infty} (h_T/\sqrt{T \ln\ln T})$. *If* $0 < \bar{d}_1 \le \bar{d}_2 < \infty$, *then the diffusion is called normal from the point of view of the upper function; otherwise, the diffusion is anomalous; we refer to it as a subdiffusion if* $\bar{d}_2 = 0$ *or a superdiffusion for* $\bar{d}_1 = \infty$.

As it was noted above, in the case of $Y_T \to Y_\infty$, $T \to \infty$, where Y_∞ is a r.v., any positive increasing function $h_T \sim g_T$, see (2.11.14) can be chosen as the upper function h_T, and then according to Definition 2.11.3, we reveal a subdiffusion. First, we present a result on the classification of diffusions based on the mean-squared displacements.

Corollary 1 ([92]) *Let* $d_1^* = \liminf_{t\to\infty} (EZ_t^2/\delta_t)$ *and* $d_2^* = \limsup_{t\to\infty} (EZ_t^2/\delta_t)$, *where* EZ_t^2 *is defined in* (2.11.5). *Then the following types of diffusions take place:*

1) *a normal diffusion for* $0 < d_1^* \le d_2^* < \infty$;

2) *a subdiffusion for* $d_2^* = 0$;

3) *a superdiffusion for* $d_1^* = \infty$.

We proved the following result.

Theorem 2.11.4 ([92][24)]) *Let Assumption* 2.11.1 *be satisfied with* $\hat{\kappa} \ne -1, -2$, *and for* $\hat{\kappa} = 0$ *it holds that* $\liminf_{t\to\infty} (\dot{\delta}_t t/\delta_t) \ge 0$. *Then the types of diffusions, as detected from the mean square displacement dynamics and the upper functions, coincide and can be determined from assertions* 1)–3) *of Corollary* 1:

24) See Theorem 5 in *Palamarchuk E.S.* On upper functions for anomalous diffusions governed by time-varying Ornstein–Uhlenbeck process // Theory of Probability & Its Applications. 2019. Vol. 64. No. 2. P. 209–228.

1) *a normal diffusion for* $0 < d_1^* \leq d_2^* < \infty$*;*

2) *a subdiffusion for* $d_2^* = 0$*;*

3) *a superdiffusion* $d_1^* = \infty$.

The cases $\hat{\kappa} = -1$ and $\hat{\kappa} = -2$ are treated separately in [92]. According to the results of the analysis, a disagreement with the types of diffusions determined on the basis of the mean-squared displacements was revealed, related to the use of a characteristic based on quadratic variations, see Theorem 2.11.3, with faster or, conversely, slower growth in time.

2.11.3 On anomalous subdiffusion modeling by a time-inhomogeneous linear stochastic differential equations in the broad sense

Assume that the equation (2.10.6) specifies a velocity process, then the displacement process and the mean-squared displacement are defined by (2.11.2) and (2.11.3) respectively. According to Definition 2.11.2, subdiffusion occurs at $D_T/T \to 0,\ T \to \infty$. If, however, the type of diffusion is determined based on a comparison of upper functions, see [92] and Definition 2.11.3, then subdiffusion takes place for $\limsup\limits_{T\to\infty}\{|Y_T|/\sqrt{T\ln\ln T}\} = 0$. Based on the results of Lemma 2.10.1 and Theorem 2.10.2, the following assertion is formulated.

Lemma 2.11.1 ([88][25]) *Let Assumption 2.10.2 be true. Then the following assertions hold*

a) *If* $\bar{h}_t t \to 0,\ t \to \infty$*, for the function* $\bar{h}_t$ *in* (2.10.10)*, then the process* Z_t *defines subdiffusion in the mean-square;*

b) *If* $\{h_t(t/\ln\ln t)\} \to 0,\ t \to \infty$*, for the function* h_t *in* (2.10.13)*, then the process* Z_t *defines subdiffusion with respect to the upper function.*

For further analysis, it is necessary to employ Assumption 2.11.1 on the change of δ_t, $t \to \infty$, the function that determines the stability rate in (2.10.6). An approach based on the study of normalized processes with the help of lemma type statements of [65] is applied. In this case, the normalizing functions have the order T (for the mean-square type subdiffusion) or $\sqrt{T\ln\ln T}$ (subdiffusion with respect to upper functions).

Theorem 2.11.5 ([88][26]) *Let Assumptions 2.10.2 and 2.11.1 be true with* $\hat{\kappa} > -2$*. If the function* $\bar{h}_t$*, defined in* (2.10.10)*, and coefficients of* (2.10.6) *satisfy*

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[25] See Assertion in *Palamarchuk E.S.* On asymptotic behavior of solutions of linear inhomogeneous stochastic differential equations with correlated inputs // Differential Equations. 2022. Vol. 58. No. 10. P. 1291–1308.

[26] See Theorem 2 in *Palamarchuk E.S.* On asymptotic behavior of solutions of linear inhomogeneous stochastic differential equations with correlated inputs // Differential Equations. 2022. Vol. 58. No. 10. P. 1291–1308.

a)

$$\frac{G_t^2 + \sigma_t^2 \bar{h}_t + tE\tilde{f}_t^2}{\delta_t^2} \to 0, \quad t \to \infty\,,$$

then the process Z_t defines subdiffusion in the mean-square;

b) *the condition that the function*

$$\frac{G_t^2 + \sigma_t^2 \bar{h}_t}{\delta_t^2} \sqrt{\frac{t}{\ln \ln t}}$$

is bounded, $t \to \infty$, and also one of the following two conditions:

$$\frac{|\tilde{f}_t|}{\delta_t} \sqrt{\frac{t}{\ln \ln t}} \to 0, \qquad a.s., t \to \infty\,,$$

or otherwise the expression

$$t\frac{E\tilde{f}_t^2}{\delta_t^2} \sqrt{\frac{t}{\ln \ln t}}\,, \quad t \to \infty\,,$$

is bounded, the process Z_t defines subdiffusion with respect to the upper function.

It was also noted in [88] that in the case of $\hat{\kappa}/\bar{\kappa} \leq -2,$ where $\hat{\kappa}$ and $\bar{\kappa}$ are constants from Assumption 2.11.1 and inequality (2.10.7), respectively, even in the simplest situation of a deterministic equation (2.10.6) with a non-trivial initial condition $z \neq 0,$ the relation $\lim\limits_{T\to\infty}\inf\{D_T/T\} > 0,$ is satisfied, i.e. there is no subdiffusion.

3 Conclusions

1. We studied stochastic linear time-varying systems as the time parameter tends to infinity.
2. For systems under controlled inputs, an analysis of optimal control problems over an infinite time horizon was carried out.
3. When investigating the problem of optimality of linear stochastic systems, we employed a methodology based on the definition of the so-called optimal stable control law as a limiting form in problem solutions on a finite planning horizon.
4. Optimality criteria are designed extending the criteria of long-run averages and their structure takes into account the impact time-varying coefficients of the control system.
5. We introduced criteria normalizations in the form of adjusted variances of cumulative disturbances. The contribution of perturbations is reflected via the squared norm of the diffusion matrix, and the adjustment factor is related to the specifics of the coefficients of the deterministic part.
6. Using the designed non-ergodic criteria for various classes of linear stochastic systems, we revealed conditions on optimality of the established stable control law.
7. In particular, it was found that when applying pathwise criteria (i.e., optimizing a.s.), unbounded growth of the corresponding normalizations is required.
8. For linear SDEs associated with optimal processes, closed form expressions are found for upper bounds that majorize sample paths with probability 1 and depend on the coefficients of the equations.
9. The estimates are generalized for the scalar case of adding multiplicative noise to the dynamics.
10. Then the considered types of linear SDEs were used in the analytical modeling of anomalous diffusions.
11. The definition of anomalous diffusion is provided on the basis of a closed form expression for mean-squared displacements.
12. It turned out that the inverse problem of determining the coefficients of underlying SDE to reproduce a given mean-squared displacement also has a solution associated with the Riccati equation.
13. A probabilistic approach is presented in which the pathwise estimates of the displacement process in the form of upper functions are derived, and the types of diffusions are revealed.

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